MAT 4872 Midterm Review Spring 2007

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Answer **all** questions carefully and give references for material used from journals. Sketch all state spaces for all transformations. You are required to work independently.

1. Let X and Y be continuous random variables with densities $f_X(x)$ and $f_Y(y)$, respectively, and suppose that X and Y are independent. Use the transformation $U = X + Y$, $V = X$, to show that the density of U satisfies the convolution equation: $f_U(u) = \int_{-\infty}^{\infty} f_X(x) f_Y(u-v)\,dv$. If X and Y are i.i.d. uniform distributions on $(0,1)$, find the distribution of $U = (X+Y)\,\mathrm{mod}1$.

2. We will examine the quadratic equation $Ax^2 + Bx + C$, where A, B and C are independent random variables, uniformly distributed on $(0,1)$.
   a. Find the distribution of $X = -2\ln B$.
   b. Find the distribution of $Y = -\ln A - \ln C$.
   c. Explain why X and Y are independent.
   d. Show that the probability that the quadratic equation have real roots is equivalent to solving $P(Y - X \geq \ln 4)$.
   e. Evaluate: $P(Y - X \geq \ln 4)$.
   f. Write a program to simulate the values of A, B and C. Your program must count the number of times that your simulated triple, (A, B, C) generate real roots to the quadratic equation. Run your program for ten thousand trials and compare the probability obtained with your answer in part e.

3. Let $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 5 \\ -1 & 5 & -4 \\ 5 & -4 & 13 \end{pmatrix}\right)$. Define $Y_1 = 5X_1 - 2X_2$ and $Y_2 = X_2 + 3X_3$. Find the moment generating function of $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$.

4. The random variables Y_1 and Y_2 are i.i.d. , both with density $f(y) = y^{-2}$ for $y \in (1, \infty)$ and zero elsewhere. Find the joint density of U_1 and U_2, where $U_1 = \frac{Y_1}{Y_1 + Y_2}$ and $U_2 = Y_1 + Y_2$. Find the marginal density of U_1.

5. A bivariate distribution is always normal if its marginals are normal. Prove or give a counterexample to this claim. Justify all steps in this solution.

Problems and solutions.

1. Let X and Y be continuous random variables with densities $f_X(x)$ and $f_Y(y)$, respectively, and suppose that X and Y are independent. Use the transformation $U = X + Y$, $V = X$, to show that the density of U satisfies the convolution equation: $f_U(u) = \int_{-\infty}^{\infty} f_X(x) f_Y(u-v)\,dv$. If X and Y are i.i.d. uniform distributions on $(0,1)$, find the distribution of $U = (X+Y)\,\mathrm{mod}\,1$.

The first part is a direct computation.

U **is a uniform distribution on** $(0,1)$**. Explain this carefully.**

2. We will examine the quadratic equation $Ax^2 + Bx + C$, where A, B and C are independent random variables, uniformly distributed on $(0,1)$.

   g. Find the distribution of $X = -2\ln B$.

   $X \square \frac{1}{2}e^{-x/2}$ **for** $x \in (0,\infty)$ **and zero elsewhere. This is an exponential distribution.**

   h. Find the distribution of $Y = -\ln A - \ln C$.

   $Y \square ye^{-y}$ **for** $y \in (0,\infty)$ **and zero elsewhere. This is a gamma distribution.**

   i. Explain why X and Y are independent.

   This is clear. Explain

   j. Show that the probability that the quadratic equation have real roots is equivalent to solving $P(Y - X \geq \ln 4)$.

   Start with $B^2 \geq 4AC$ **and take logs of both sides.**

k. Evaluate: $P(Y - X \geq \ln 4)$

The region $\{(x,y)|0 \leq x < \infty, 0 \leq y < \infty\}$ **gets mapped into the region** $\{(u,v)|0 < v < \infty,\ v \geq u\}$ **by the transformation:** $U = Y - X$ **and** $V = Y$**.**

$$f(u) = \begin{cases} \frac{2}{9}e^{u/2} & \text{for } u \in (-\infty, 0) \\ \frac{1}{3}\left(u + \frac{2}{3}\right)e^{-u} & \text{for } u \in (0, \infty) \end{cases}$$

$$P(Y - X \geq \ln 4) = P(U \geq \ln 4) = \int_{\ln 4}^{\infty} \frac{1}{3}\left(u + \frac{2}{3}\right)e^{-u}du = \frac{5}{36} + \frac{1}{6}\ln 2$$
$$\approx 0.2544$$

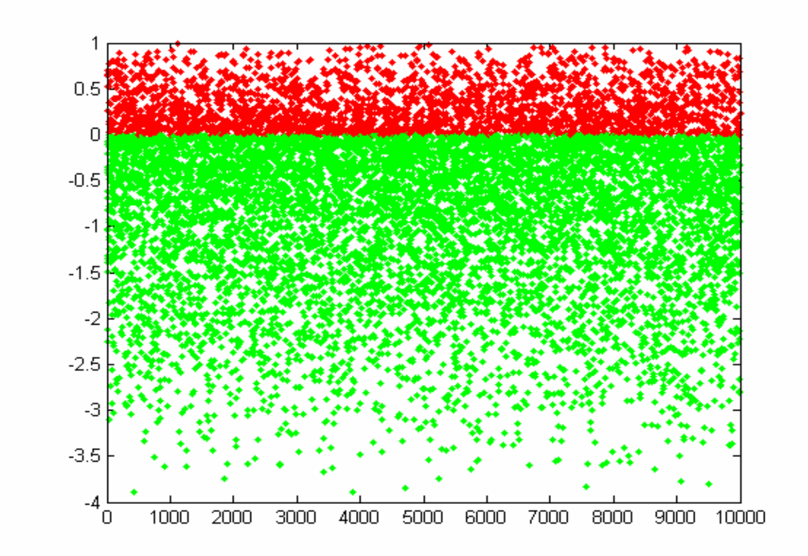
l. Write a program to simulate the values of A, B and C. Your program must count the number of times that your simulated triple, (A, B, C) generate real roots to the quadratic equation. Run your program for ten thousand trials and compare the probability obtained with your answer in part e.

The matlab program below simulates the triple (A, B, C)**; finds the probability of real roots; and gives a graphical illustration of the distribution of the real roots in red and the imaginary roots in green.**

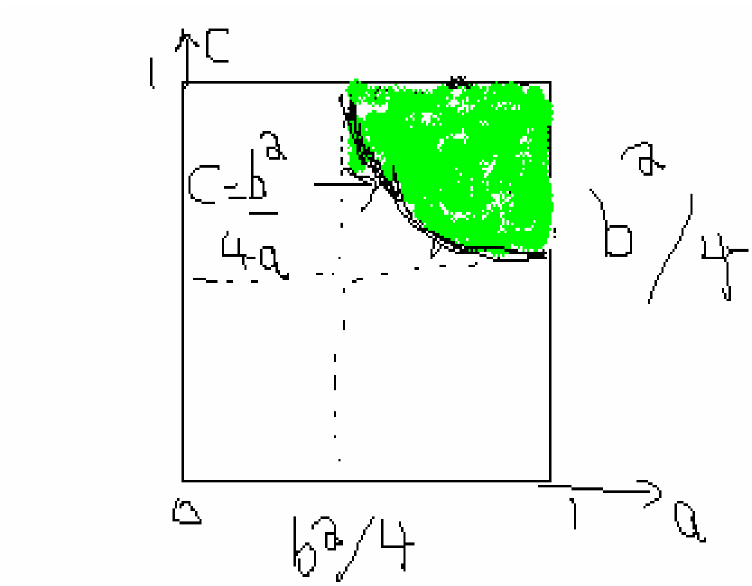
```
function qroot(n)
counter=0;
for i=1:n
A=rand; B=rand; C=rand; D=B^2-4*A*C;
if (D>=0)
counter=counter+1;
plot(i,D,'r.')
hold on
else
plot(i,D,'g.')
hold on
end
```

```
end
probability = counter/n
end
```

Probability= 0.2579 for 10,000 triples, which is close to the theoretical value.



Here is an alternative way to solve the above problem.
Solution II.



For imaginary roots: $c > b^2/4a$ (See the shaded region above), if b is fixed, then $b^2/4a < c < 1$, $b^2/4 < a < 1$ from the above diagram. Clearly $0 < b < 1$, therefore the probability of real roots is:

$$1-\int_0^1\int_{b^2/4}^1\int_{b^2/4a}^1 dc\,da\,db=\frac{5}{36}+\frac{1}{6}\ln 2$$

The above solution is much more nicer and it can be extended to consider all quadratics. If we consider a,b and c to be uniform on $(-1,1)$, then imaginary roots can be found when a and c have the same sign. In the above analysis we consider the case $a>0$ and $c>0$. The joint density function of a,b and c is $1/8$ on the cube of side 2 units centered around the origin.

Now the probability of real roots is:

$$1-2\int_{-1}^1\int_{b^2/4}^1\int_{b^2/4a}^1 \frac{1}{8}dc\,da\,db=\frac{41}{72}+\frac{1}{12}\ln 2\approx .6272067..$$

A simulation run by matlab with the program qroot1 gives:
qroot1(100000000)
probability = 0.62718992000000

```
function qroot1(n)
counter=0;
for i=1:n
A=2*rand-1; B=2*rand-1; C=2*rand-1;
D=B^2-4*A*C;
if (D>=0)
counter=counter+1;
end
end
probability = counter/n
end
```

3. Let $\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}\sim N\left(\begin{pmatrix}1\\3\\7\end{pmatrix},\begin{pmatrix}2&-1&5\\-1&5&-4\\5&-4&13\end{pmatrix}\right)$. Define $Y_1=5X_1-2X_2$ and $Y_2=X_2+3X_3$. Find the moment generating function of $\begin{pmatrix}Y_1\\Y_2\end{pmatrix}$.

$$M_{\binom{y_1}{y_2}}(\underline{t})=\exp\left(\underline{t}'\underline{\mu}+\frac{1}{2}\underline{t}'\Sigma\underline{t}\right)=\exp(45t_1^2+84t_1t_2+49t_2^2+24t_2-t_1)$$

4. The random variables Y_1 and Y_2 are i.i.d. , both with density $f(y) = y^{-2}$ for $y \in (1, \infty)$ and zero elsewhere. Find the joint density of U_1 and U_2, where $U_1 = \dfrac{Y_1}{Y_1 + Y_2}$ and $U_2 = Y_1 + Y_2$. Find the marginal density of U_1.

$$f_u(u) = \begin{cases} \dfrac{1}{2\left(1-u^2\right)} & \text{for } u \in \left(0, \dfrac{1}{2}\right) \\ \dfrac{1}{2u^2} & \text{for } u \in \left[\dfrac{1}{2}, 1\right] \end{cases}$$

5. A bivariate distribution is always normal if its marginals are normal. Prove or give a counterexample to this claim. Justify all steps in this solution.

Consider the joint density $f(x,y) = \dfrac{\left(1 + xye^{-0.5\left(x^2+y^2-2\right)}\right)}{2\pi} e^{-0.5\left(x^2+y^2\right)}$, where $x \in \square$ and $y \in \square$, for a counterexample.