Math 4872 Final Examination Review Spring 2008
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Answer **all** questions carefully. Your answers must include the appropriate sketches of the state spaces. All matlab codes must be handed in.

1. a. The speed of a molecule in a uniform gas at equilibrium is a random variable X with density function $f(x) = ax^2 e^{-bx^2}$, $x \in [0,\infty)$ where $b = m/2kT$ and k, T, m are constants. Define $Y = \frac{1}{2}mX^2$. Show that $Y \sim \frac{1}{(kT)^{3/2}\Gamma(3/2)}\sqrt{y}e^{-y/kT}$, $y \in [0,\infty)$ and find the $E(Y)$.

$Y = \frac{1}{2}mX^2$, $\Rightarrow x = \sqrt{2y/m}$, $\frac{dx}{dy} = \frac{1}{\sqrt{2my}}$;

$f_Y(y) = a\left(\sqrt{2y/m}\right)^2 e^{-b\left(\sqrt{2y/m}\right)^2}\left(1/\sqrt{2my}\right) = \left(a\sqrt{2y}/m^{3/2}\right)e^{-2by/m}$,

comparison with the standard γ($\alpha = 3/2$, $\beta = kT$) distribution gives

$a\sqrt{2y}/m^{3/2} = \frac{1}{(kT)^{3/2}\Gamma(3/2)}$. $\Rightarrow$ $Y \sim \frac{1}{(kT)^{3/2}\Gamma(3/2)}\sqrt{y}e^{-y/kT}$, $y \in [0,\infty)$.

$E(Y) = \alpha\beta = \frac{3}{2}kT$.

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b. Let X_1 and X_2 be i.i.d. uniformly distributed over $(0,1)$. Find the p.d.f. of each of the following: i. $X_1 X_2$ ii. X_1/X_2

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i. Let $U = X_1 X_2$ and $V = X_1$, $0 < V \le 1$, and $0 < U/V \le 1$,

$$J = \begin{vmatrix} 0 & 1 \\ 1/v & -u/v^2 \end{vmatrix} = -1/v \Rightarrow |J| = 1/v. \quad f_U(u) = \int_u^1 1/v\,dv = -\ln u,\ 0 < u \le 1.$$

ii. $U = X_1/X_2$, $V = X_2$, $0 < V \le 1$, $0 < UV \le 1$, $J = \begin{vmatrix} 0 & 1 \\ 1/v & -u/v^2 \end{vmatrix} \Rightarrow |J| = v$.

$$f_U(u) = \begin{cases} \int_0^1 v\,dv = 1/2,\ 0 < u \le 1 \\ \int_0^{1/u} v\,dv = \frac{1}{2u^2},\ 1 < u < \infty \end{cases}$$

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2. Let $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 14 & 0 & 9 \\ 0 & 3 & 0 \\ 9 & 0 & 6 \end{pmatrix} \right)$.

Find $E[X_1, X_2 \mid X_3 = 4]$ and factor the above distribution as a product of two normal distributions, where the mean of one of the factors is $E[X_1, X_2 \mid X_3 = 4]$.

$$E[X_1, X_2 \mid X_3 = 4] = \mu_{1|2} = \mu_1 + \sum_{11} \sum_{22}^{-1} (x_2 - \mu_2) = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$f(x_1, x_2, x_3) = \frac{1}{\pi\sqrt{6}} e^{-\frac{1}{2}\left(2(x_1-7)^2 + \frac{1}{3}(x_2-1)^2\right)} . \frac{1}{2\sqrt{3\pi}} e^{-\frac{1}{12}(x_3-2)^2}$$

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3. Let $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix} \right)$. Do the following:

i. Write out the p.d.f. of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, and plot its graph using matlab.

ii. Find the constant density contour that contains 50% of the probability and plot it using matlab.

iii. Using matlab, simulate 100 pairs of random values (x_1, x_2), where $x_1 \in X_1$ and $x_2 \in X_2$ and plot them on the same axes as your contour in part (iii).

iv. Modify your code in part (iii) to count the number of points contained in your density contour. How does your result compare with the theoretical prediction?

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i. $f(x_1, x_2) = \dfrac{1}{\pi\sqrt{6}} e^{-\frac{1}{2}\left(\frac{2}{3}x_1^2 - \frac{2\sqrt{2}}{3}x_1(x_2-2) + \frac{4}{3}(x_2-2)^2\right)}$

```
[x1,x2]=meshgrid(-5:.02:5);
p=-0.5*((2/3)*x1.^2-((2/3)*(sqrt(2)))*x1.*(x2-2)+(4/3)*(x2-2).^2);
f=(1/(pi*sqrt(6))).*exp(p);
mesh(x1,x2,f)
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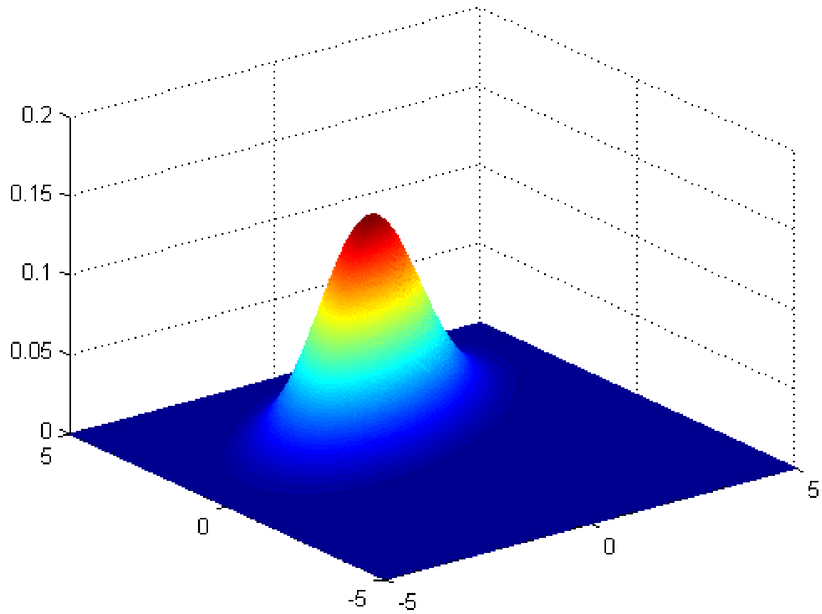
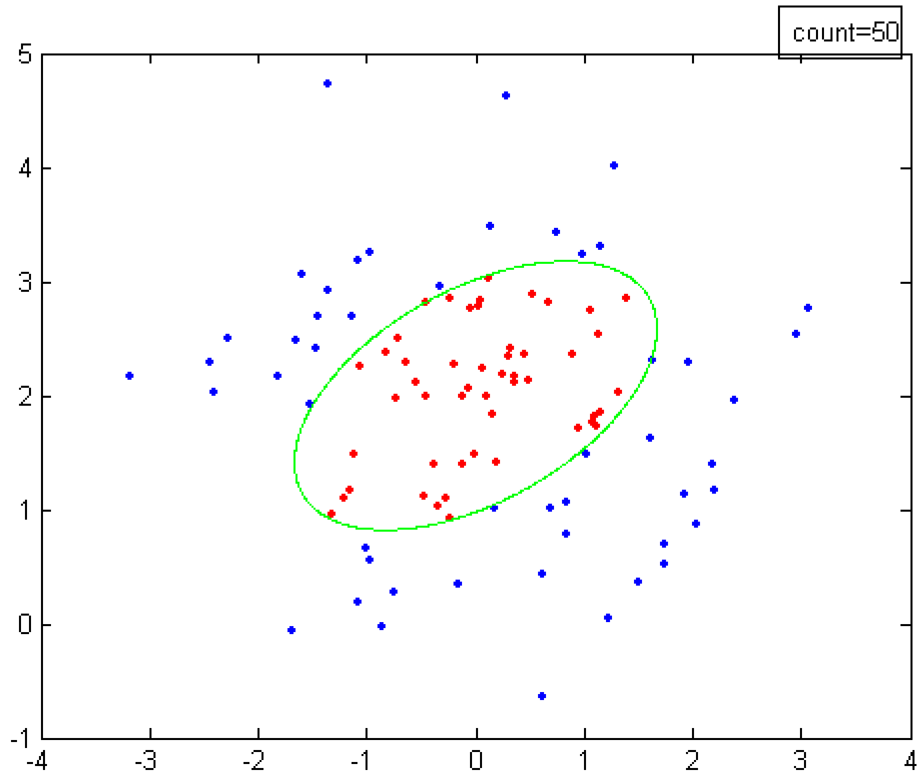


ii. $\frac{2}{3}x_1{}^2 - \frac{2\sqrt{2}}{3}x_1\left(x_2-2\right) + \frac{4}{3}\left(x_2-2\right)^2 \le \chi_2^2\left(1/2\right) = 1.39$

iii, iv.




```
x1=(sqrt(2))*randn(1,100);
x2=2+randn(1,100);
      count=0;
for i=1:100
j=(2/3)*x1(i).^2-(2*sqrt(2)/3)*x1(i).*(x2(i)-2)+(4/3)*(x2(i)-2).^2;
     if j<= 1.39
         count=count+1;
         plot(x1(i),x2(i),'.r')
    hold on
     else count=count;
         plot(x1(i),x2(i),'.b')
    hold on
     end

end
count
[x,y]=meshgrid(-6:.01:6);
j=(2/3)*x.^2-(2*sqrt(2)/3)*x.*(y-2)+(4/3)*(y-2).^2;
contour(x,y,j-1.39,[0,0],'g')
```

4. A clinical psychologist wished to compare three methods for reducing hostility levels in university students. A certain psychological test(HLT) was used to measure the degree of hostility. High scores on this test indicate great hostility. Eleven students that obtained high and nearly equal scores were used in this experiment. Five were selected at random from among the eleven problem cases and treated by method A. Three were taken at random from the remaining six students and treated by method B. The other three students were treated by method C. All treatments continued throughout the semester. Each student was given the HLT test again at the end of the semester, with the results shown in the table below. Do the data provide sufficient evidence to indicate that at least one of the methods of treatment produces a mean student response different from the other methods? Give bounds for the attained significance level. What would you conclude at the $\alpha = 0.05$ level of significance? You need to show all calculations. Generate the ANOVA table both by hand and by Matlab or Maple. Submit all results and codes used.

Method A	Method B	Method C
73	54	79
83	74	95
76	71	87
68		
80		

From the Anova, $F = 5.148 > 4.46$ so we reject H_0.

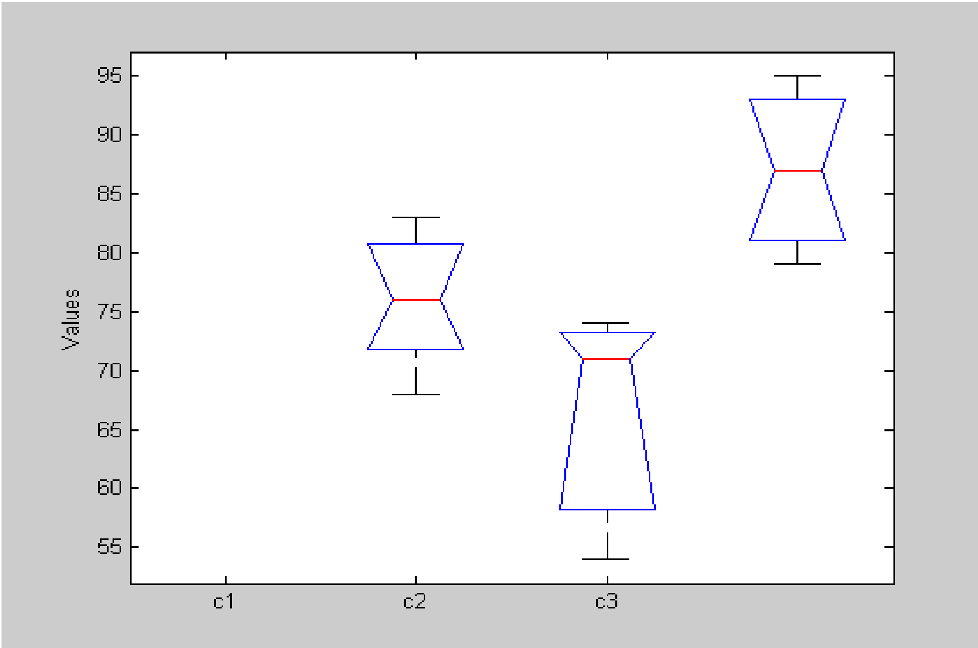
Note: $4.46 < 5.148 < 6.06 \Rightarrow .025 < p - value < .05$

Matlab Code:

```
>> A=[73 83 76 68 80 54 74 71 79 95 87];
>> b=({'c1', 'c1', 'c1', 'c1', 'c1', 'c2','c2', 'c2', 'c3','c3','c3'});
>> [b,tbl,stats]=anova1(A,b)
```

ANOVA Table

Source	SS	df	MS	F	Prob>F
Groups	641.88	2	320.939	5.15	0.0365
Error	498.67	8	62.333		
Total	1140.55	10			



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5. A moment-generating function approach can be used to find the small sample distribution for the Wilcoxon test statistic W under H_0. First recall that for a discrete random variable X taking on values $a_1, a_2, \ldots, a_n$ with probabilities $p_1, p_2, \ldots p_n$, respectively, the m.g.f. of X would be $M_x(t) = E\left[e^{tX}\right] = \sum_{i=1}^{n} e^{ta_i} . p_i$.

   i. Show that the m.g.f. of the random variable $j \cdot S_j$ in 9.26 (see page 398 of your text) is $\cosh(jt)$.

   ii. Write the general expression for the m.g.f. of $W^{/}$ in (9.26).

   iii. Apply your expression to the case where $n = 3$ to find the m.g.f. of $W^{/}$ and, from it, the probability mass function of W.

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i. $E\left(e^{t.jS_j}\right) = \frac{1}{2}e^{-jt} + \frac{1}{2}e^{jt} = \cosh(jt)$.

ii. $E\left(e^{t\sum jS_j}\right) = \prod_{j=1}^{n} E\left(e^{t.jS_j}\right) = \prod_{j=1}^{n}\left(\frac{1}{2}e^{-jt} + \frac{1}{2}e^{jt}\right)$

iii. For $n=3$, the mgf is $\frac{1}{8}\left(e^{-6t} + e^{-4t} + e^{-2t} + 2e^{0t} + e^{2t} + e^{4t} + e^{6t}\right)$.

$$p(w) = \begin{cases} 1/8 & w = -6, -4, -2, 2, 4, 6 \\ 1/4 & w = 0 \end{cases}$$
