Math 2580 Final Examination Review Spring 2008
By Professors S. Singh and S. Han

1. A and B are 4×4 matrices with $\det(A) = -2$ and $\det(B) = 3$. Give the numerical value of the following expressions:
   i. $\det\left(AB\right)$ ii. $\det\left(5A^{-1}\right)$ iii. $\det\left(B^T\right)$ iv. $\det\left(B^T A^{-1}\right)$ v. $\det\left(B^{10}\right)$

2. Let $\mathrm{H} = \left\{ \begin{bmatrix} x_1 \\ x_1 - x_2 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$.
   a. Prove that H is a subspace of $\mathbb{R}^3$.
   b. Find a basis of H.
   c. Describe the span of H and plot it in $\mathbb{R}^3$ using matlab.

3. Consider the system of equations:

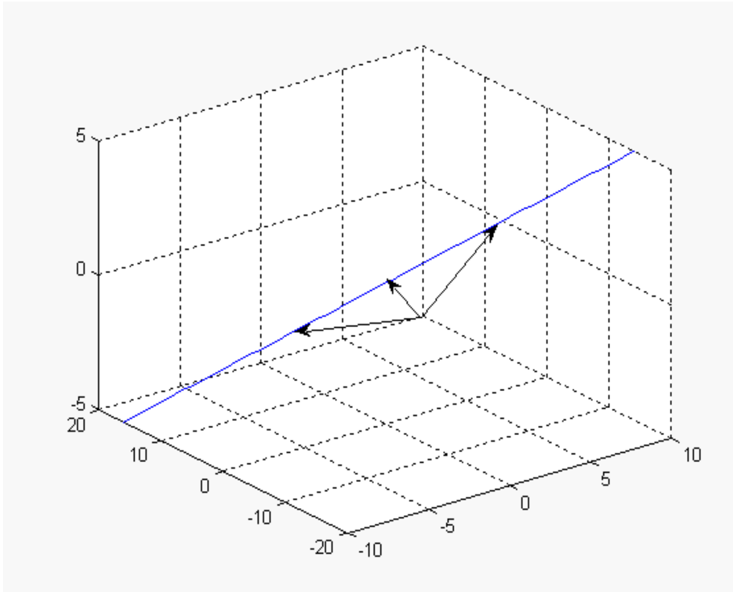
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= -1 \\ x_1 + 3x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 3x_3 &= -2 \end{aligned}$$

   a. Rewrite the system in the form $A\underline{x} = B$.
   b. Find the inverse of A and use it to solve the above system. Credit will not be given for any other method.

4. Let the matrix $\mathrm{M} = \begin{bmatrix} 1 & 2 & -2 & 1 & 0 & 0 \\ 3 & 6 & -5 & 4 & 0 & 0 \\ 1 & 2 & 0 & 3 & 0 & 0 \end{bmatrix}$:
   a. Find a basis of the column space of M and state its dimension.
   b. Find a basis of the null space of M and state its dimension.

5. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.
   a. Find the eigenvalues and eigenvectors of A.
   b. Rewrite A in the form $A = PDP^{-1}$, and use this to find A^5.

6. a. Consider **three vectors** in $\mathbb{R}^3$ touching a line that does not pass through the origin. If the vectors are not multiples of each other, prove that they are linearly dependent. See the diagram below (Diagram not drawn to scale).



b. In part (a), if we replace **three vectors** by **two vectors** with the above property, will the two vectors be linearly dependent? Explain your answer.

7. Give a 2×2 matrix R that will rotate the vector $\underline{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ through an angle of $\frac{\pi}{4}$ radians. Show that $\|R\underline{v}\| = \|\underline{v}\|$ and the angle between $R\underline{v}$ and $\underline{v}$ is $\frac{\pi}{4}$ radians.

8. Evaluate the following determinant:

$$\begin{bmatrix} 1 & -5 & 0 & 0 & 0 \\ -5 & 8 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 \\ 0 & 0 & \frac{1}{3} & \pi & 1 \end{bmatrix}$$

9. a. If A and B are invertible matrices, prove that $(A+B)^{-1} = A^{-1} + B^{-1}$ is false by giving a counter example. Show why your example satisfies the requirement.

b. Find C^{-1}, where $C = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 3 & 15 \end{bmatrix}$.

10. Decide whether the following sets of vectors are linearly dependent. In each case explain your reasoning.

a. $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

b. Let u_1, u_2, u_3 and u_4 be vectors belonging to $\mathbb{R}^4$ such that $u_1 = \pi u_2$.

11. A and P are invertible $n \times n$ matrices such that $B = PAP^{-1}$.

a. Prove that $\det A = \det B$.
b. Prove that $\det(B - \lambda I_n) = \det(A - \lambda I_n)$, where λ is some constant.
c. Prove that $B^n = PA^nP^{-1}$, for $n = 1, 2, 3\ldots$.

12. Can this augmented system $\begin{bmatrix} \pi a & b & 1 & 0 \\ w & t & 3 & 0 \\ \sqrt{t^4} & 5 & 5 & 0 \end{bmatrix}$ be made inconsistent for any choice of a, b, w and t from the set of real numbers?

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Solutions.

1. i. $\det(AB) = -6$

   ii. $\det(5A^{-1}) = -312.5$

   iii. $\det(B^T) = 3$

   iv. $\det(B^T A^{-1}) = -1.5$

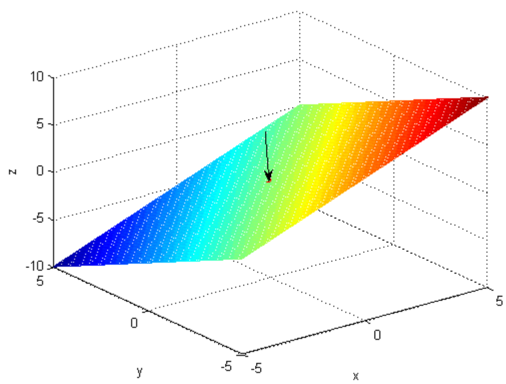
   v. $\det(B^{10}) = 59049$

2. a. Verify that the three conditions are satisfied.

   b. The basis vectors of H are $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

   c. The span of H is $\mathbb{R}^2$. The basis vectors of H, spans the plane $z = x - y$.

```
[x,y]=meshgrid(-5:.1:5,-5:.1:5);
z=x-y;
mesh(x,y,z)
hold on
plot3(0,0,0,'r+')
```



3. a. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$

   b. $A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 2/3 & 0 & -1/3 \end{bmatrix}$, $\underline{x} = A^{-1}B = \begin{bmatrix} -5 \\ 2 \\ 0 \end{bmatrix}$

4. a. $\text{Col}M = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 0 \end{bmatrix} \right\}$, dim = 2.

b. $\text{Nul}M = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, dim = 4

5. a. The eigenvalues and their corresponding eigenvector:
$\lambda = -1, \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $\lambda = 5, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

b. **Solution I.**

$$P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix},\ A^5 = PD^5P^{-1}.$$

Solution II.

$$P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix},\ A^5 = PD^5P^{-1}.$$

6. b. The two vectors will be linearly independent.

7. $R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$, $\|R\underline{v}\| = \|\underline{v}\| = 5$

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9. a. There are many examples. One such pair is: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

b. $C^{-1} = \begin{bmatrix} 9 & -3 & 1 \\ -34 & 13 & -4 \\ 8 & -3 & 1 \end{bmatrix}$

10. a. Linearly dependent.
    b. Linearly dependent.

11. No.