New York City College of Technology
Department of Mathematics

MAT 2440 Final Exam Review Problems[1]

1. Determine whether $(\neg p \wedge (p \vee q)) \to q$ is a tautology.

2. Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.

3. Let $P(x)$, $Q(x)$ and $R(x)$ be the statements "x is an engineer," "x is smart," and "x is vain," respectively, where the domain consists of all people. Translate each of these statements into English.

(a) $\forall x \neg(P(x) \wedge Q(x))$

(b) $\exists x(R(x) \wedge \neg Q(x)) \to \exists y P(y)$

4. (Follow-up to previous problem.) Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, and $R(x)$.

(a) All smart people are vain.

(b) No engineers are smart.

(c) There is a person that is both vain and an engineer.

5. Negate the following statements so that the negation appears only within the predicates.

(a) $\forall x \exists y P(x, y)$

(b) $\exists y(Q(y) \wedge \forall x \neg R(x, y))$

6. Determine whether the following arguments are valid. If the argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

(a) All dogs are mammals. Spike is a dog. Therefore, Spike is a mammal.

(b) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

(c) Quinn likes rock bands. Quinn likes the Punch Brothers. Therefore, the Punch Brothers is a rock band.

(d) Consider the argument form in Table 1.

$$\begin{array}{l} p \wedge q \\ p \to r \\ q \to s \\ \hline \therefore r \wedge s \end{array}$$

Table 1: Question 6(d).

7. Show that, for every integer n, n^2 is even if and only if n is even.

8. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

(a) a proof by contraposition.

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(b) a proof by contradiction.

9. Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 - x + 2$.

(a) Find $f(S)$ for $S = \{-1, 0, 1, 2\}$.

(b) Is f one-to-one? Explain.

(c) Is f onto? Explain.

10. Find the first four terms a_0, a_1, a_2, a_3 of each sequence.

(a) $a_n = n^3 + \dfrac{2}{n+1}$ for $n \geq 0$

(b) $a_n = (-2)^n$ for $n \geq 0$

11. Find the terms a_1, a_2, a_3 for the sequence given by the following recurrence relation: $a_n = 2n + a_{n-1}$ for $n \geq 1$ and $a_0 = 2$.

12. Find the values of each of the sums.

(a) $\sum_{j=0}^{4}(1 + (-2)^j)$

(b) $\sum_{i=1}^{3}\sum_{j=1}^{3}(i - j)$

13. Write the pseudocode for an algorithm that takes a list of n integers and produces as output the sum of the numbers in the list.

14. Write the pseudocode for an algorithm that finds both the largest and smallest integers in a finite sequence of integers.

15. Show that $x^3 + 3x + 1$ is $\Theta(x^3)$. You must show what witnesses you obtained in your calculations.

16. Find all pairs of functions in this list that are of the same order:

$$n^2 + \log n, 2^n + 3^n, 100n^3 + n^2, n^2 + 2^n, n^2 + n^3, 3n^3 + 3^n.$$

17. Give a big-O estimate for the number of times the following algorithm prints something:

```
procedure FUN1(n : integers)
    s := 0
    for i := n  to n^2 do
        for j := 1  to n do
            print i + j
        end for
    end for
end procedure
```

18. Evaluate the quantity $(-55 \pmod{14} + 21 \pmod{14}) \pmod{14}$.

19. Convert:

(a) 703 from decimal to base 7.

(b) $(ABBA)_{16}$ from hexadecimal to octal.

20. Find an inverse of $a = 10$ modulo $m = 17$.

21. Solve the congruence using the Euclidean Algorithm: $5x \equiv 3 \pmod{23}$.

22. Decrypt the message QBOKD DRSC MVKCC SC that was encoded using the shift cipher $f(p) = (p + 10) \pmod{26}$.

23. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n. Prove that $P(n)$ is true for $n \geq 1$.

24. Prove that for every positive integer n, $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3$.

25. Prove that 6 divides $n^3 - n$ whenever n is a positive integer.

Answers:

1. .

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$(\neg p \wedge (p \vee q)) \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Therefore $(\neg p \wedge (p \vee q)) \rightarrow q$ is a tautology.

2. .

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	T	F	F	T	F	T
T	F	F	F	T	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	T	F	T	T

Therefore $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.

3. **(a)** Everyone is not both an engineer and smart.

(b) If there is someone who is vain and not smart, then there exists someone who is an engineer.

4. **(a)** $\forall x(Q(x) \rightarrow R(x))$

(b) $\neg \exists x(P(x) \wedge Q(x)) \equiv \forall x \neg (P(x) \wedge Q(x)) \equiv \forall x(\neg P(x) \vee \neg Q(x))$

(c) $\exists x(P(x) \wedge R(x))$

5. **(a)** $\exists x \forall y \neg P(x, y)$

(b) $\forall y(\neg Q(y) \vee \exists x R(x, y))$

6. **(a)** Valid. It uses the argument in Table 2.

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Table 2: Answer for 6(a).

(b) Valid. It uses the argument in Table 3.

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Table 3: Answer for 6(b).

$p \to q$
q
$\therefore p$

Table 4: Answer for 6(c).

(c) Invalid. It uses the argument in Table 4.

(d) Valid. The fact that $p \wedge q$ is true means that p and q are true. It follows that, since $p \to r$ is true, r is true. Similarly, s is true, so $r \wedge s$ is true.

7. Suppose n is even. Then there exists an integer k such that $n = 2k$. It follows that $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. So n^2 is even. Suppose n is odd. Then there exists an integer k such that $n = 2k+1$. It follows that $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. So n^2 is odd. By contraposition, if n^2 is even, then n is even.

8. (a) Let n be odd. Then there exists an integer k such that $n = 2k+1$. It follows that $n^3 + 5 = (2k+1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2(4k^3 + 6k^2 + 3k + 3)$. So $n^3 + 5$ is even. By contraposition, if $n^3 + 5$ is odd, then n is even.

(b) Suppose n is odd and $n^3 + 5$ is odd. Then there is an integer k such that $n = 2k+1$. Then (using the same calculations from part (a)) $n^3 + 5 = 2(4k^3 + 6k^2 + 3k + 3)$, which shows that $n^3 + 5$ is even. This is a contradiction, so either n is even (the statement we want) or $n^3 + 5$ is even (the contrapositive of what we want). It follows that if $n^3 + 5$ is odd, then n is even.

9. (a) For $f(x) = x^2 - x + 2$, we have

$$\begin{aligned} f(-1) &= (-1)^2 - (-1) + 2 = 4 \\ f(0) &= 0^2 - 0 + 2 = 2 \\ f(1) &= 1^2 - 1 + 2 = 2 \\ f(2) &= 2^2 - 2 + 2 = 4. \end{aligned}$$

Hence $f(S) = \{2, 4\}$.

(b) The function f is not one-to-one, because $f(0) = f(1)$.

(c) The function f is not onto. The graph of f is a concave up parabola with vertex, say, (a, b). Then any value of y less than b is not an image. In particular, $f(x) = 0$ has no real solutions.

10. (a)

$$\begin{aligned} a_0 &= 0^3 + \frac{2}{0+1} = 2 \\ a_1 &= 1^3 + \frac{2}{1+1} = 2 \\ a_2 &= 2^3 + \frac{2}{2+1} = 8 + \frac{2}{3} = \frac{26}{3} \\ a_3 &= 3^3 + \frac{2}{3+1} = 27 + \frac{2}{4} = \frac{110}{4} = \frac{55}{2} \end{aligned}$$

(b)

$$\begin{aligned} a_0 &= (-2)^0 = 1 \\ a_1 &= (-2)^1 = -2 \\ a_2 &= (-2)^2 = 4 \\ a_3 &= (-2)^3 = -8 \end{aligned}$$

11.

$$a_1 = 2 \cdot 1 + a_0 = 2 + 2 = 4$$
$$a_2 = 2 \cdot 2 + a_1 = 4 + 4 = 8$$
$$a_3 = 2 \cdot 3 + a_2 = 6 + 8 = 14$$

12. **(a)**

$$\begin{aligned}
&\sum_{j=0}^{4}(1 + (-2)^j) \\
=&(1 + (-2)^0) + (1 + (-2)^1) + (1 + (-2)^2) + (1 + (-2)^3) + (1 + (-2)^4) \\
=&1 + 1 + 1 - 2 + 1 + 4 + 1 - 8 + 1 + 16 \\
=&16
\end{aligned}$$

(b)

$$\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{3}(i - j) \\
=&\sum_{i=1}^{3}[(i-1) + (i-2) + (i-3)] \\
=&[(1-1) + (1-2) + (1-3)] + [(2-1) + (2-2) + (2-3)] + [(3-1) + (3-2) + (3-3)] \\
=&0 - 1 - 2 + 1 + 0 - 1 + 2 + 1 + 0 \\
=&0
\end{aligned}$$

13.

```
procedure SUM(a_1, ..., a_n: integers)
    sum := 0
    for j := 1 to n do
        sum := sum + a_j
    end for
    return sum {sum of a_1 to a_n}
end procedure
```

14.

```
procedure MAXMIN(a_1, ..., a_n: integers)
    max := a_1
    min := a_1
    for j := 2 to n do
        if a_j > max then
            max := a_j
        if a_j < min then
            min := a_j
    end for
    return {max, min} {returns max and min}
end procedure
```

15. For $x > 1$, we have that $x^3 \leq x^3 + 3x + 1 \leq 5x^3$. So the function is $\Theta(x^3)$.

16. $(2^n + 3^n, 3n^3 + 3^n)$ and $(100n^3 + n^2, n^2 + n^3)$

17. The algorithm prints something $n(n^2 - n + 1) = n^3 - n^2 + n$ times. So the number of times it prints something is $O(n^3)$.

18. $(-55 \pmod{14} + 21 \pmod{14}) \pmod{14} = (-55 + 21) \pmod{14} = -34 \pmod{14} = 8$.

19. **(a)**

$$\begin{aligned}703 &= 7 \cdot 100 + 3\\100 &= 7 \cdot 14 + 2\\14 &= 7 \cdot 2 + 0\\2 &= 7 \cdot 0 + 2\end{aligned}$$

Thus $703 = (2023)_7$.

(b) We have

$$\begin{aligned}(ABBA)_{16} &= (1010\ \ 1011\ \ 1011\ \ 1010)_2\\&= (001\ \ 010\ \ 101\ \ 110\ \ 111\ \ 010)_2\\&= (125672)_8.\end{aligned}$$

20. We have

$$\begin{aligned}17 &= 1 \cdot 10 + 7\\10 &= 1 \cdot 7 + 3\\7 &= 2 \cdot 3 + 1\\3 &= 3 \cdot 1.\end{aligned}$$

Then $\gcd(17, 10) = 1$ and

$$\begin{aligned}1 &= 7 - 2 \cdot 3\\&= 7 - 2 \cdot (10 - 1 \cdot 7) = 3 \cdot 7 - 2 \cdot 10\\&= 3 \cdot (17 - 1 \cdot 10) - 2 \cdot 10 = 3 \cdot 17 - 5 \cdot 10.\end{aligned}$$

Hence $-5 \cdot 10 \equiv 1 \pmod{17}$, that is, $12 \cdot 10 \equiv 1 \pmod{17}$. So 12 is the inverse of 10 modulo 17.

21. We have:

$$\begin{aligned}23 &= 4 \cdot 5 + 3\\5 &= 1 \cdot 3 + 2\\3 &= 1 \cdot 2 + 1\\2 &= 2 \cdot 1.\end{aligned}$$

Therefore $\gcd(23, 5) = 1$ and

$$\begin{aligned}1 &= 3 - 1 \cdot 2\\&= 3 - 1 \cdot (5 - 1 \cdot 3) = 2 \cdot 3 - 1 \cdot 5\\&= 2 \cdot (23 - 4 \cdot 5) - 1 \cdot 5\\&= 2 \cdot 23 - 9 \cdot 5.\end{aligned}$$

Then $-9 \cdot 5 \equiv 1 \pmod{23}$. By multiplying both sides by 3, we obtain that $5 \cdot (-27) \equiv 3 \pmod{23}$. So $x \equiv -27 \pmod{23}$, that is, $x \equiv 19 \pmod{23}$.

22. We replace the letters in the message QBOKD DRSC MVKCC SC by numbers:

16 1 14 10 3   3 17 18 2   12 21 10 2 2   18 2.

Now we decode each number using $f^{-1}(p) = (p - 10) \pmod{26}$:

$$6\ 17\ 4\ 0\ 19 \quad 19\ 7\ 8\ 18 \quad 2\ 11\ 0\ 18\ 18 \quad 8\ 18.$$

Translating this message back to letters gives "GREAT THIS CLASS IS."

23. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for the positive integer n. When $n = 1$,

$$1^2 = 1 = \frac{6}{6} = \frac{1(1+1)(2 \cdot 1 + 1)}{6},$$

so $P(1)$ is true. Assume that $P(k)$ is true for some $k \geq 1$. Then

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}. \end{aligned}$$

So $P(k+1)$ is true. By induction, $P(n)$ is true for all $n \geq 1$.

24. Let $P(n)$ be the statement that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for the positive integer n. When $n = 1$, we have

$$1 \cdot 2 = 2 = \frac{6}{3} = \frac{1(1+1)(1+2)}{3},$$

so $P(1)$ is true. Assume $P(k)$ is true for some $k \geq 1$. Then

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)[(k+1)+1] &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \\ &= \frac{(k+1)[(k+1)+1][(k+1)+2]}{3}. \end{aligned}$$

Then $P(k+1)$ is true. By induction, $P(n)$ is true for all $n \geq 1$.

25. Let $P(n)$ be the statement "6 divides $n^3 - n$" for a positive integer n. When $n = 1$, we have $1^3 - 1 = 0$. Since 6 divides 0, we have that $P(1)$ is true. Assume $P(k)$ is true for some $k \geq 1$. Then

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + (3k^2 + 3k) \\ &= (k^3 - k) + 3k(k+1). \end{aligned}$$

By induction hypothesis, $k^3 - k$ is divisible by 6. Since either k or $k+1$ is even, we obtain that $3k(k+1)$ is divisible by 6. So $(k^3 - k) + 3k(k+1)$ is divisible by 6, that is, 6 divides $(k+1)^3 - (k+1)$. Then $P(k+1)$ is true. By induction, $P(n)$ is true for all $n \geq 1$.