Arithmetic | Algebra

Edition 3.0

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ii

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Preface

This book is intended to be used in conjunction with WebWork online homework system as a textbook for the math remedial courses MAT 0630 and MAT 0650 at New York City College of Technology, CUNY.

The authors chose to write this document to provide a customized open-source text for the remedial students at City Tech. The book can be easily adapted to fit the needs of remedial students at other CUNY colleges.

The book consists of short chapters, addressing essential concepts necessary to master to successfully proceed to credit-level math courses. Each chapter provides solved examples, one unsolved "Exit Problem" and is supplemented by its own WebWork assignment. The content in the book and WebWork are also aligned to prepare students for the CEAFE exam.

We sincerely thank Professor Ariane Masuda for her insightful edits, and Professor Nadia Kennedy for her helpful comments and edits on some of the content of this book. We also thank the students Joe Nathan Abellard (Computer Engineering) and Ricky Santana (Math Education) who edited an earlier draft of this work.

Contents

Chapter 1

Integers

1.1 Integers

We begin with a brief review of arithmetic with integers i.e. $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.

$$\xleftarrow{\qquad} \underset{-5}{\quad} \underset{-4}{\quad} \underset{-3}{\quad} \underset{-2}{\quad} \underset{-1}{\quad} \underset{0}{\quad} \underset{1}{\quad} \underset{2}{\quad} \underset{3}{\quad} \underset{4}{\quad} \underset{5}{\quad} \xrightarrow{\qquad}$$
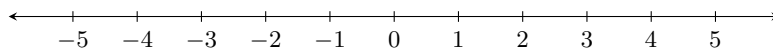
Any number has a weight and a sign.

> The **magnitude** (or **weight**) of a number is the distance it is from 0 on the number line.

Example 1.1. For example, the weight of -5 is 5 and the weight of 7 is 7.

> Two numbers are **opposites** if, on the number line, they are on opposite sides of zero, but the same distance away from zero.

So, -5 is the *opposite* of 5, and 7 is the opposite of -7 and so on.

Addition

We can add two numbers with the help of a number line .

Example 1.2. Adding two positive numbers: For example to add $3 + 4$, we start with 3 on the number line then move 4 units to the right. We land at 7 which is our answer.

Move 4 units to the *right*
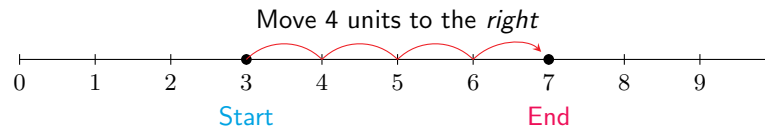
So, $3 + 4 = 7$. Notice that the answer has the same sign as the signs of 3 and 4 (both positive) and its weight comes from adding the weights of 3 and 4.

You always move to the right when you add a positive number

Example 1.3. Adding two negative numbers: For example, $-10 + (-4)$ means you are adding a debt of $\$4$ to an already existing debt of $\$10$. So, we start at -10 on the number line and move 4 units to the left, to land at -14, which is the answer.

You always move to the left when you add a negative number (a debt)
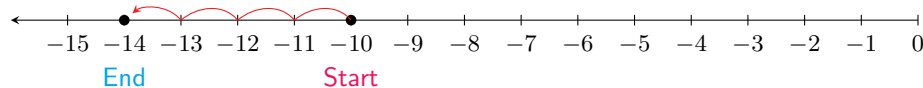
So, $-10 + (-4) = -14$. Notice that the answer has the same sign as the signs of -10 and -4 (both negative) and its weight comes from adding the weights of -10 and -4.

To add numbers of opposite signs, that is, a positive and a negative number, we can also use the number line. For example, to perform $10 + (-4)$, we start at 10 on the number line and then move 4 units to the left. We land at 6, which is the answer. Think of $10 + (-4)$ as having $\$10$ and adding a $\$4$ debt. Because we are adding a debt, we move to the left on the number line!

Move 4 units to the *left*
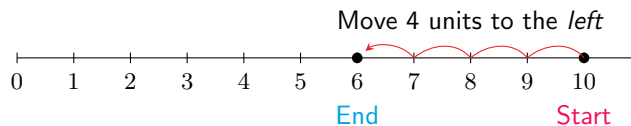
So, $10 + (-4) = 6$. Notice that the answer has the same sign as the sign of 10 (positive) because it is the number of larger weight, and its weight comes from finding the difference of the weights of 10 and -4.

Notice that because we were adding two numbers of opposite signs, the answer ended up being the difference in weight (6) along with the sign of the number of larger weight (positive).

Example 1.4. Adding two numbers of opposite signs: For example, $3 + (-7)$. We start at 3 and move toward 7 units to the left, and we land at -4, which is our answer.

Move 7 units to the *left*

-5 $\quad$ -4 $\quad$ -3 $\quad$ -2 $\quad$ -1 $\quad$ 0 $\quad$ 1 $\quad$ 2 $\quad$ 3 $\quad$ 4 $\quad$ 5 $\quad$ 6 $\quad$ 7 $\quad$ 8

End $\qquad\qquad\qquad\qquad\qquad\qquad$ Start
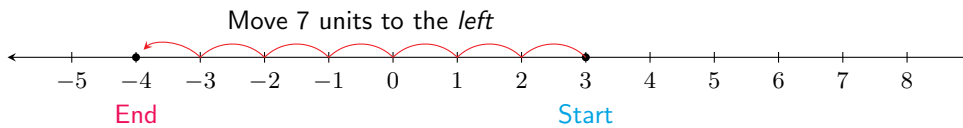
So, $3 + (-7) = -4$. Notice that the answer has the same sign as the signs of -7 (negative) because it is the number of larger weight, and its weight comes from finding the difference of the weights of 3 and -7.

Example 1.5. Adding opposites: We start at -5 on the number line and jump to the right 5 units to land finally at 0. So $-5 + 5 = 0$.

Move 5 units to the *right*

-5 $\quad$ -4 $\quad$ -3 $\quad$ -2 $\quad$ -1 $\quad$ 0 $\quad$ 1 $\quad$ 2 $\quad$ 3 $\quad$ 4 $\quad$ 5 $\quad$ 6 $\quad$ 7 $\quad$ 8

Start $\qquad\qquad\qquad\qquad\qquad$ End
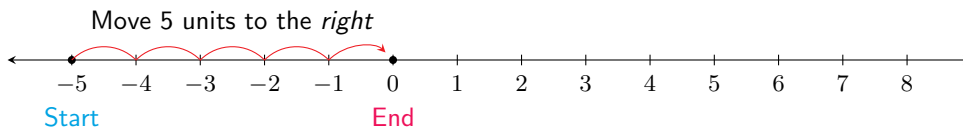
Note 1.6. Two opposite numbers are called a zero-pair, because adding them always results in 0.

So, -5 and 5 are a zero-pair.

Adding Integers

1. To **add two numbers of the same sign**, add their weights and place it after the sign.

2. To **add two numbers of opposite signs**, find the difference of their weights and place it after the sign of the number with the greater weight.

Example 1.7. Add:

a) $-8 + 19 = 11$

b) $-8 + 4 = -4$

c) $6 + (-9) = -3$

d) $7 + (-2) = 5$

e) $(-4) + (-7) = -11$

f) $8 + 7 = 15$

Note 1.8. While we can add in any order: $4 + 2 = 2 + 4$, it is sometimes convenient to add up all of the negative numbers and add up all of the positive numbers, and then add the results. There are also times when it is best to notice certain simplifications if the numbers are added in a different order.

    For example

$$-5 + 4 + 5 + (-8) = -5 + (-8) + 4 + 5 \text{ (by reordering)}$$

so,

$$-5 + 4 + 5 + (-8) = -5 + (-8) + 4 + 5 = -13 + 9 = -4$$

    We could have also simplified this by noting that (-5) and 5 are zero-pair, so we are left with $4 + (-8)$ which is -4:

Example 1.9. We can calculate

$$(-4) + (-5) + 7 + (-3) = (-4) + (-5) + (-3) + 7 = (-12) + 7 = -5$$

We could have simplified this by noting that (-4) and (-3) make -7, and -7 and 7 are zero-pair, so the total is -5:

Subtraction (as Addition of the Opposite)

Once we know how to add numbers, we are set to subtract numbers because subtraction is nothing but addition of the opposite. That is, subtracting $8 - 3$ (which reads: subtracting 3 from 8) is the same as $8 + (-3)$ (which reads: Adding -3 to 8.).

Example 1.10. So, $8 - 3 = 8 + (-3)$, and, we can use the rules of adding two numbers of opposite signs to find out that the answer is 5. We can also use the number line. We start at 8 and move 3 units to the left (adding -3 is adding a debt, so we move to the left). So $8 - 3 = 5$.
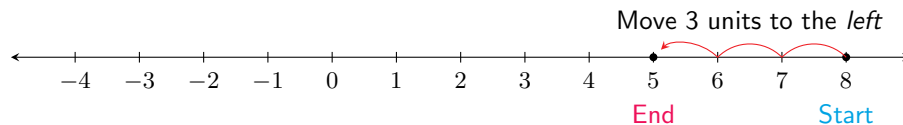
Example 1.11. To calculate $3 - 7$ we first rewrite it as an addition problem. $3 - 7 = 3 + (-7)$. We can either use the number line, or the rules of adding two numbers of opposite signs. And, so, $3 - 7 = 3 + (-7) = -4$.

Example 1.12. To calculate $-4 - 1$, we first rewrite it as an addition problem. $-4 - 1 = -4 + (-1)$. We can either use the number line, or the rules of adding two numbers of the same signs. If we want to use the rules, both numbers are negative, so our answer will be negative, and, adding the weights of -4 and -1 is 5. So, $-4 - 1 = -4 + (-1) = -5$

On the number line, we start at -4 and move 1 unit to the left, to land on -5 which is our answer.
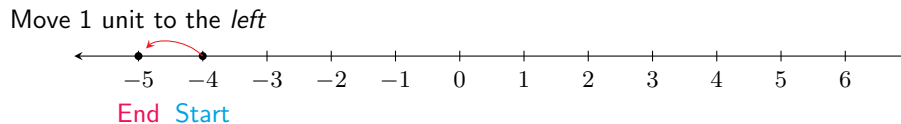
Changing the subtractions to additions in this way is particularly useful when adding or subtracting several numbers (because we can add in any order).

Example 1.13.

$$- 3 - 7 + 5 + 7 + 13 - 6 - (-9)$$

$$= -3 + (-7) + 5 + 7 + 13 + (-6) + 9$$

$$= -3 + (-7) + (-6) + 5 + 7 + 13 + 9$$

$$= 18$$

$$-3 - 7 + 5 + 7 + 13 - 6 - (-9)$$

$$= -3 + (-7) + 5 + 7 + 13 + (-6) + 9$$

$$= 5 + 13$$

$$= 18$$

Remark 1.14. Warning: The symbol $"-"$ is used in two different ways. When it is between two expressions, it means subtract (e.g., $3 - 4$). Otherwise, it means 'opposite' or 'negative' (e.g., $-3 + 4$). So in the expression $-4 - (-3)$, the first and last $"-"$ means opposite and the one in the middle one means subtract. The importance of understanding this can not be overestimated.

Multiplication and Division of Positive Numbers

Multiplication of integers is adding in the sense that $3 \times 4 = 4 + 4 + 4$.

To multiply larger numbers, it is better to use the usual scheme of multiplication. For example:

Example 1.15. Let us multiply 152 by 34. We will for convenience sake put the smaller number on the bottom (though it is not necessary). We have

```
        1   5   2
    ×       3   4
        6   0   8
 +  4   5   6   0
    5   1   6   8
```

And division is the opposite of multiplication in the sense that to compute $45 \div 9$ is to find a number so that when we multiply by 9 we get 45. We run through our multiplication tables (which are hopefully in our head) to discover that 5 does the trick: $5 \times 9 = 45$ so that $45 \div 9 = 5$. We will discuss division from a different point of view when we discuss fractions.

To divide larger numbers, we can use long division. For example, let's divide 3571 by 11:

Example 1.16.

$$
\begin{array}{r}
324 \\
11\ \overline{\big)\ 3571} \\
-33 \\
\hline
271 \\
-22 \\
\hline
51 \\
-44 \\
\hline
7 \quad = \text{remainder}
\end{array}
$$

Multiplication involving negative numbers

Multiplication is a little tricky to understand without the notion of distribution (discussed later). We will begin with noting again what it means to multiply a number by a positive number: So if we want to compute $4 \cdot (-7)$ we note

$$4 \cdot (-7) = (-7) + (-7) + (-7) + (-7) = -28.$$

Note that since $4 \cdot 7 = 28$, $4 \cdot (-7) = -(4 \cdot 7)$. We can multiply positive numbers in any order: $4 \cdot 7 = 7 \cdot 4$. The same is true of positive and negative numbers:

$$(-7) \cdot 4 = 4 \cdot (-7) = -(4 \cdot 7) = -28.$$

Example 1.17. $5 \cdot (-12) = -(5 \cdot 12) = -60$ and $(-3) \cdot (-2) = -(3 \cdot (-2)) = -(-(3 \cdot 2)) = 6$

Remark 1.18. So the size of the product of two numbers is the product of their sizes. The sign is positive if the signs are the same and negative if they are different.

Remark 1.19. Two quantities right next to each other, with no symbol between them (except for parentheses around either or both numbers), has an implicit multiplication. For example, $3(2) = 3 \times 2$.

Example 1.20. Multiply:

a) $(-5)(-8) = 40$

b) $(-6) \cdot 7 = -42$

c) $4 \cdot 12 = 48$

d) $(-3)(-6) \cdot 4(-3) = 18 \cdot 4(-3) = 72(-3) = -216$ (multiplying from left to right)

e) $(-3)(-5) \cdot 4(-2) = (-3) \cdot 4 \cdot (-5)(-2) = -12 \cdot 10 = -120$ (since we can multiply in any order it is convenient to see that $-5 \cdot -2 = 10$.)

Exponents of integers

Recall that a positive exponent represents the number of times a number is multiplied by itself.

Example 1.21. Evaluate:

a) $5^2 = 5 \cdot 5 = 25$

b) $(-4)^3 = (-4) \cdot (-4) \cdot (-4) = 16 \cdot (-4) = -64$

c) $(-7)^1 = -7$

d) $-2^4 = -2 \cdot 2 \cdot 2 \cdot 2 = -16$. **Note:** The exponent here is for 2 not -2!

e) $(-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 9 \cdot (-3) \cdot (-3) = -27 \cdot (-3) = 81$

f) $(-2)^5 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) = -32$

Exponent rules in details will be further discussed in Chapter 5.

Division involving negative numbers

Division is just a question of knowing multiplication and therefore has the same rule: The size of the quotient of two numbers is the quotient of the sizes. The sign is positive if the signs are the same and negative if they are different.

Example 1.22. Divide:

a) $(-42) \div 7 = -6$

b) $81 \div (-9) = -9$

c) $(-35) \div (-7) = 5$

d) $14 \div 2 = 7$

e) $0 \div 5 = 0$. **Note** When dividing 0 by any number, the answer is always 0.

f) $-10 \div 0 =$ undefined.

Note 1.23. Any number divided by 0 is undefined!

> **Multiplying and Dividing Integers**
> Consider two numbers at a time.
>
> 1. If the signs of the two numbers are the same, then the sign of the answer is positive.
>
> 2. If the signs of the two numbers are different, then the sign of the answer is negative.

1.2   The Order of Operations

What is the meaning of the expression '3 times 4 plus 5'. Some will answer 17 while others may answer 27. Why? To take the ambiguity out, we can write $(3 \times 4) + 5 = 17$ and $3 \cdot (4 + 5) = 27$, where we must first evaluate the quantity in parentheses. Since it can be somewhat cumbersome to write a lot of parentheses, there is an important convention or agreement that if we just write $3 \times 4 + 5$ we mean $(3 \times 4) + 5$. That is, in the absence of parentheses, we should multiply before we add. This is part of what is called *The Order of Operations*. This must be remembered.

Definition 1.24 (The Order of Operations)**.** When evaluating an expression involving addition, subtraction, multiplication and division which has no parentheses or exponents, we first perform, from left to right, all of the multiplications and divisions. Then, from left to right, the additions and subtractions. If there are parts of the expression set off by parentheses, what is within the parentheses must be evaluated first.

Remark 1.25. Subtraction can be turned into addition and then addition can be done in any order, not necessarily from left to right. This explains why addition and subtraction come together in the order of operations. There will be a similar statement for multiplication and division but will be postponed until fractions are discussed.

'PE(MD)(AS)' is an easy way to remember the order of operations. This means that the order is: Parentheses, Exponents (this will be incorporated later), Multiplication and Division (taken together from left to right), and finally, Addition and Subtraction (taken together from left to right).

Let us try a few problems.

Example 1.26. Evaluate:

a) $3 + 2(3 + 5) = 3 + 2(8) = 3 + 16 = 19$

b) $3 - 2(-4 + 7) = 3 - 2(3) = 3 - 6 = -3$

c) $-3 - 4 - 2(-2 \cdot 6 - 5) = -3 - 4 - 2(-12 - 5) = -3 - 4 - 2(-17) = -3 - 4 - (-34) = -3 - 4 + 34 = 27$

d) $-(3-(-6))-(1-4\cdot(-5)+4)=-(3+6)-(1-(-20)+4)=-9-$
$(1+20+4)=-9-25=-9+(-25)=-34$

e) $-2(-14\div7+7)=-2(-2+7)=-2(5)=-10$

f) $-3(-2\cdot7-(-5)(4)\div2)=-3(-14-(-20)\div2)=-3(-14-(-10))=$
$-3(-4)=12$

g) $6\div2\times3=3\times3=9$ **Note:** $6\div2\times3\neq6\div6=1$

h) $-2(3-1)^2-(8-2^2)\div4=-2(2)^2-(8-4)\div4=-2(4)-4\div4=-8-1=-9$

Exit Problem

Evaluate: $(3^3+5)\div4-4(7-2)$

Chapter 2

Fractions

2.1   Fractions

In our previous section we identified integers: $0, \pm 1, \pm 2, \pm 3, \dots$. To this set, now we will add all ratios of integers with non-zero denominators, like $\dfrac{2}{7}, \dfrac{-11}{17} \dots$. We call this the set of **rational numbers**. Any rational number looks like $\dfrac{p}{q}$ where p and q are integers and q is not 0.

Just as we are able to perform arithmetic operations with integers we can also perform arithmetic operations with rational numbers (fractions). The two types of fractions we will encounter are called proper and improper:

Proper fractions have value less than 1, for example $\dfrac{2}{5}$ and $\dfrac{1}{8}$. Observe that for these fractions the numerator is less than the denominator.

Improper fractions have value greater than or equal to 1, for example $\dfrac{7}{6}$ and $\dfrac{3}{2}$. For these fractions the numerator is greater than the denominator.

Each fractional value can have many different, equivalent forms, for example $1 = \dfrac{2}{2} = \dfrac{-5}{-5} = \dots$. In order to determine whether two fractions are equivalent we can use the fundamental principle of fractions.

The Fundamental Principle of Fractions

$$\frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12}$$

That is, as long as you multiply both numerator and denominator by the **same** number, the fraction value does not change, and you obtain equivalent fractions.

Example 2.1. Write a fraction that is equivalent to $\dfrac{3}{5}$.

Begin with our original fraction $\dfrac{3}{5}$ and apply the fundamental principle of fractions to get

$$\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10}.$$

Example 2.2. Simplify the fraction $\dfrac{15}{35}$.

Begin with our original fraction and apply the fundamental principle of fractions in reverse to get

$$\frac{15}{35} = \frac{3 \cdot \cancel{5}}{7 \cdot \cancel{5}} = \frac{3}{7}.$$

Multiplying Fractions

We multiply numerators together and denominators together:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Example 2.3. The product of these two fractions is done as follows:

$$\frac{14}{3} \cdot \frac{9}{7}$$

$$\frac{14 \cdot 9}{3 \cdot 7} = \frac{2 \cdot \cancel{7} \cdot \cancel{3} \cdot 3}{\cancel{3} \cdot \cancel{7}} = \frac{6}{1} = 6$$

> ### The Reciprocal of a Fraction
> The reciprocal of a fraction $\dfrac{p}{q}$ is the fraction formed by switching the numerator and denominator, namely $\dfrac{q}{p}$

Example 2.4. Find the reciprocal of the given fraction:

a) The reciprocal of $\dfrac{3}{5}$ is $\dfrac{5}{3}$.

b) The reciprocal of $\dfrac{-2}{7}$ is $\dfrac{7}{-2} = \dfrac{-7}{2} = -\dfrac{7}{2}$.

c) The reciprocal of $\dfrac{1}{8}$ is $\dfrac{8}{1} = 8$.

d) The reciprocal of $4 = \dfrac{4}{1}$ is $\dfrac{1}{4}$.

Dividing Fractions

We multiply the first fraction by the **reciprocal** of the second:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

Example 2.5. The quotient of these two fractions is found as follows:

$$\frac{8}{3} \div \frac{4}{5}$$

$$\frac{8}{3} \div \frac{4}{5} = \frac{8}{3} \cdot \frac{5}{4} = \frac{8 \cdot 5}{3 \cdot 4}$$

$$= \frac{2 \cdot \cancel{4} \cdot 5}{3 \cdot \cancel{4}} = \frac{10}{3}$$

Adding Fractions (with same denominators)

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

Subtracting Fractions (with same denominators)

$$\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$$

Example 2.6. Add $\dfrac{3}{5} + \dfrac{1}{5}$

$$\frac{3}{5} + \frac{1}{5} = \frac{3 + 1}{5} = \frac{4}{5}$$

Adding or Subtracting Fractions (with unlike denominators)

Adding (or subtracting) fractions with unlike denominators requires us to first find a common denominator. The **LCD** or **least common denominator** is the smallest number that both denominators evenly divide. Once we rewrite each of our fractions so their denominator is the LCD, we may add or subtract fractions according to the above properties.

> **Finding the LCD**
>
> Step 1. Make a list of (enough) multiples of each denominator.
>
> Step 2. Identify the lowest common multiple. If you can't see one, then your lists in Step 1. need to be expanded.

To be able to add or subtract fractions, we need to go one more step: Once you've identified the LCD, rewrite both fractions (by multiplying both numerator and denominator by the appropriate same number) to get the LCD as denominator.

Example 2.7. Find the LCD and then add and simplify $\dfrac{3}{12} + \dfrac{5}{8}$.

Let us first find the LCD by following our procedure.

Step 1. Make a list of (enough) multiples:
$$8 : 8, 16, 24, 32, \ldots$$
$$12 : 12, 24, 36, 48, \ldots$$

Step 2. LCD: 24, $8 \cdot 3 = 24$, $12 \cdot 2 = 24$

Step 3. Rewrite each fraction using the LCD:

$$\frac{3}{12} = \frac{3 \cdot 2}{12 \cdot 2} = \frac{6}{24}$$

and

$$\frac{5}{8} = \frac{5 \cdot 3}{8 \cdot 3} = \frac{15}{24}$$

Now we are ready to add our fractions

$$\frac{6}{24} + \frac{15}{24} = \frac{21}{24}$$

simplifying yields

$$\frac{21}{24} = \frac{\cancel{3} \cdot 7}{\cancel{3} \cdot 8} = \frac{7}{8}$$

Example 2.8. Find the LCD and then subtract and simplify $\dfrac{1}{9} - \dfrac{3}{5}$.

Let us first find the LCD by following our procedure.

Step 1. Make a list of (enough) multiples:
$9 : 9, 18, 27, 36, 45, 54, 63, \dots$
$5 : 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, \dots$

Step 2. LCD: 45

Step 3. Rewrite each fraction using the LCD:

$$\frac{1}{9} = \frac{1 \cdot 5}{9 \cdot 5} = \frac{5}{45}$$

and

$$\frac{3}{5} = \frac{3 \cdot 9}{5 \cdot 9} = \frac{27}{45}$$

Now we are ready to subtract our fractions, but, first, we rewrite the subtraction as addition of the opposite:

$$\frac{1}{9} - \frac{3}{5} = \frac{1}{9} + \left(-\frac{3}{5}\right) = \frac{5}{45} + \left(-\frac{27}{45}\right) = \frac{5 + (-27)}{45} = \frac{-22}{45}$$

Writing an Improper Fraction as a Mixed Number

1. Divide the numerator by the denominator.

2. If there is a remainder, write it over the denominator.

Example 2.9. Write $\dfrac{42}{5}$ as a mixed number.

We begin by dividing the numerator 42 by the denominator 5 to get 8, with a remainder of 2. Our mixed number is $8\dfrac{2}{5}$.

Writing a Mixed Number as an Improper Fraction

1. Multiply the whole number and the denominator then add the numerator. Use the result as your new numerator.

2. The denominator remains the same.

Example 2.10. Write the mixed number $3\dfrac{5}{6}$ as an improper fraction.

$$3\frac{5}{6} = 3 \quad \frac{5}{6} = \frac{23}{6}$$

1. Multiply the denominator by the whole number.

2. Add this result to the numerator.

3. Set this new numerator 23 over the denominator of 6.

We multiply the whole number 3 and the denominator 6 to get 18. Next, we add to this the numerator 5 to get 23. This is our new numerator and our improper fraction becomes $\dfrac{23}{6}$.

Addition and Subtraction of Mixed Numbers

To add (or subtract) mixed numbers, we can convert the numbers into improper fractions, then add (or subtract) the fractions as we saw in this chapter.

Example 2.11. Subtract $7 - 2\dfrac{3}{8}$.

First we convert $2\dfrac{3}{8} = \dfrac{19}{8}$. Then, we rewrite the subtraction operation as addition of opposite:

$$7 - \frac{19}{8} = 7 + \left(-\frac{19}{8}\right) = \frac{7}{1} + \left(-\frac{19}{8}\right) = \frac{56}{8} + \left(-\frac{19}{8}\right) = \frac{56 + (-19)}{8} = \frac{37}{8} = 4\frac{5}{8}.$$

Also, we can keep mixed fractions and mixed fraction, and, add (or subtract) the integer parts together and the fraction parts together.

Example 2.12. Add $7\frac{3}{4} + 3\frac{1}{5}$.

Here, we add $7 + 3 = 10$ and $\frac{3}{4} + \frac{1}{5} = \frac{15}{20} + \frac{4}{20} = \frac{19}{20}$.

And, our final answer is $10\frac{19}{20}$. Note that $\frac{19}{20}$ is a proper fraction, so, our work is done. But, if our answer ended up with an improper fraction, we would have had to make the conversion to write the answer in simplified form.

Multiplying and Dividing of Mixed Numbers

Be careful when multiplying mixed numbers. You must first convert them to improper fractions and use the rules for multiplying fractions to finish your problem.

Example 2.13. Multiply $2\frac{3}{5}$ and $3\frac{1}{2}$.

Begin by rewriting each mixed number as an improper fraction: $2\frac{3}{5} = \frac{13}{5}$ and $3\frac{1}{2} = \frac{7}{2}$. Now we proceed by multiplying the fractions

$$\frac{13}{5} \cdot \frac{7}{2} = \frac{13 \cdot 7}{5 \cdot 2} = \frac{91}{10}.$$

We can now write the result (if we wish) as a mixed number: $9\frac{1}{10}$.

Example 2.14. Divide $\left(1\frac{4}{5}\right) \div \left(1\frac{1}{2}\right)$.

Begin by rewriting each mixed number as a improper fraction: $1\frac{4}{5} = \frac{9}{5}$ and $1\frac{1}{2} = \frac{3}{2}$. Now we proceed by dividing the fractions

$$\frac{9}{5} \div \frac{3}{2} = \frac{9}{5} \cdot \frac{2}{3} = \frac{\cancel{9} \cdot 2}{5 \cdot \cancel{3}} = \frac{3 \cdot 2}{5 \cdot 1} = \frac{6}{5} = 1\frac{1}{5}$$

Exit Problem

Evaluate: $\frac{3}{4} - 1\frac{5}{6}$

Chapter 3

Decimal Numbers

Consider the number 23.7456. Each digit occupies a 'place'. The 2 is in the *tens* place, the three in the *ones* place, the 7 in the *tenths* place, the 4 in the *hundredths* place, the 5 in the *thousandths* place, and the 6 in the *ten-thousandths* place. Why? Because:

$$23.7456 = 2 \cdot 10 + 3 \cdot 1 + 7 \cdot \frac{1}{10} + 4 \cdot \frac{1}{100} + 5 \cdot \frac{1}{1000} + 6 \cdot \frac{1}{10,000}$$

Rounding

Rounding is associated with cutting or truncating a number, and that the rounding compensates for the lost tail of the number. For example, to round a given number to the nearest tenth, we look one digit to the right of the tenths place (the hundredths place) and if it is greater than or equal to 5, we add one to the tenths place and remove all the digits to the right, otherwise we leave the tenths place as it is and remove all the digits to the right.

Example 3.1. Round:

a) 234.45 rounded to the nearest ten is 230.

b) 45.6 rounded to the nearest ones (whole number) is 46.

c) 34.555 rounded to the nearest tenth is 34.6.

d) 34.54 rounded to the nearest tenth is 34.5.

e) 34.95 rounded to the nearest tenth is 35.0.

f) 34.554 rounded to the nearest hundredth is 34.55.

g) 56.7874778 rounded to the nearest ten-thousandth is 56.7875.

Adding and Subtracting Decimal Numbers

To add decimals, we line up the decimal points, and wherever there is a missing digit, we fill it in with a zero. For example, add 45.23 and 2.3:

$$
\begin{array}{ccccc}
4 & 5 & . & 2 & 3 \\
+ & 2 & . & 3 & 0 \\
\hline
4 & 7 & . & 5 & 3
\end{array}
$$

Subtracting is similar. To subtract 45.23 from 2.3 we first note that the answer should be negative and proceed to subtract 2.3 from 45.23:

$$
\begin{array}{ccccc}
4 & 5 & . & 2 & 3 \\
- & 2 & . & 3 & 0 \\
\hline
4 & 2 & . & 9 & 3
\end{array}
$$

So, the answer of $2.3 - 45.23 = -42.93$

Example 3.2. Add:

a) $2.4 + 32.032 = 34.432$

b) $3.44 + 12.035 = 15.475$

c) $34.3 - 0.05 = 34.25$

d) $6.3 - 9.72 = -3.42$

Multiplying and Dividing Decimal Numbers

Multiplying and Dividing Decimal Numbers by $10, 100, 1000, \ldots$

We first look at the special multiplication of decimals by $10, 100, 1000, \ldots$

$12.415 \times 10 = 124.15$
$12.415 \times 100 = 1241.5$
$12.415 \times 1000 = 12415$

When we multiply by 10 we move the decimal point to the right one place (because 10 has one decimal place). Multiplying by 100 moves the decimal point two places (because 100 has two decimal places), etc.

$12.415 \div 10 = 1.2415$
$12.415 \div 100 = 0.12415$
$12.415 \div 1000 = 0.012415$

When we divide by 10 we move the decimal point to the left one place (because 10 has one decimal place). Dividing by 100 moves the decimal point to the left two places (because 100 has two decimal places), etc.

10^n notation

$10 = 10^1$
$100 = 10 \times 10 = 10^2$
$1000 = 10 \times 10 \times 10 = 10^3$

Notice that the exponent of 10 in 10^n notation reflects the number of zeros! So, $10000 = 10^4$ (4 zeros, exponent is 4) and $100,000 = 10^5$, ...

> **Multiplying by** 10^n
> Multiplying a decimal number by 10^n moves the decimal
> place n spots to the right. For example:
>
> $$5.435 \times 10 = 54.35$$
> $$5.435 \times 100 = 543.5$$
> $$5.435 \times 10000 = 54350$$

Multiplying Decimal Numbers

To multiply two decimal numbers, we multiply as if there is no decimal point, then place a decimal point as described in the next example.

Example 3.3. Multiply 5.4 by 1.21.

$$
\begin{array}{rcccc}
  &   & 1 & 2 & 1 \\
\times &   &   & 5 & 4 \\
\hline
  &   & 4 & 8 & 4 \\
  & 6 & 0 & 5 & 0 \\
\hline
  & 6 & 5 & 3 & 4 \\
\end{array}
$$

Now, to write out the answer, we notice that there are two digits after the decimal point in the first number 1.21, and one digit after the decimal point in the second number 5.4. The product then should have 3 digits after the decimal point. So, $5.4 \times 1.21 = 6.534$.

Example 3.4. Multiply 3.72 by 13.

$$
\begin{array}{rcccc}
  &   & 3 & 7 & 2 \\
\times &   &   & 1 & 3 \\
\hline
  & 1 & 1 & 1 & 6 \\
  & 3 & 7 & 2 & 0 \\
\hline
  & 4 & 8 & 3 & 6 \\
\end{array}
$$

Now, to write out the answer, we notice that there are two digits after the decimal point in 3.72 while 13 has no decimal part. The product then should have 2 digits after the decimal point: $3.72 \times 13 = 48.36$.

Dividing Decimal Numbers

Dividing a decimal number is a lot like dividing a whole number, except you use the position of the decimal point in the dividend to determine the decimal places in the result.

Example 3.5. a) $6.5 \div 2$

Here, 6.5 is called the **dividend** and 2 is called the **divisor**.

Divide as usual:

If the divisor does not go into the dividend evenly, add zeros to the right of the last digit in the dividend and continue until either the remainder is 0, or a repeating pattern appears. Place the position of the decimal point in your answer directly above the decimal point in the dividend.

$$
\begin{array}{r}
3.25 \\
2 \overline{\smash{)}\, 6.5} \\
-6 \\
\hline
0\,5 \\
-4 \\
\hline
10 \\
-10 \\
\hline
0
\end{array}
$$

b) $55.318 \div 3.4$

If the divisor in not a whole number, move the decimal point in the divisor all the way to the right (to make it a whole number). Then move the decimal point in the dividend the same number of places.

In this example, move the decimal point one place to the right for the divisor from 3.4 to 34. Therefore, also move the decimal point one place to the right for the dividend, from 55.318 to 553.18.

$$
\begin{array}{r}
16.27 \\
34 \enclose{longdiv}{553.18} \\
-34 \\
\hline
213 \\
-204 \\
\hline
9\,1 \\
-6\,8 \\
\hline
2\,38 \\
-2\,38 \\
\hline
0
\end{array}
$$

Converting Decimals to Fractions

To convert a decimal to a fraction is as simple as recognizing the place of the right most digit.

Example: Note that in the number 2.45, the right most digit 5 is in the hundredths place so $2.45 = \dfrac{245}{100} = \dfrac{49}{20}$ or $2\dfrac{9}{20}$.

Example 3.6. Here are a few more examples:

a) $1.2 = \dfrac{12}{10} = \dfrac{6}{5}$ or $1\dfrac{1}{5}$

b) $0.0033 = \dfrac{33}{10,000}$

c) $0.103 = \dfrac{103}{1000}$

Converting Fractions to Decimals

To convert a fraction to a decimal you simply perform long division.

Example 3.7. Convert the given fraction to a decimal:

a) $\dfrac{4}{5} = 4 \div 5 = 0.8$

b) $3\dfrac{4}{5} = 3 + 4 \div 5 = 3.8$

c) $\dfrac{13}{2} = 6\dfrac{1}{2} = 6 + 1 \div 2 = 6.5$

d) (round to the nearest tenth) $\dfrac{3}{7} = 3 \div 7 = 0.42857 \cdots \approx 0.4$

Converting Decimals to Percents and Percents to Decimals

"Percent" comes from Latin and means per hundred. We use the sign % for percent. For example, if you know that 25% of the students speak Spanish fluently, it means that 25 of every 100 students speak fluent Spanish. Presented as fraction, it would be $\dfrac{25}{100}$ and as a decimal 0.25.

Example 3.8. Convert the given percent to fraction then to a decimal:

a) 17% is $\dfrac{17}{100} = 0.17$.

b) 31% is $\dfrac{31}{100} = 0.31$.

c) 23.44% is $\dfrac{23.44}{100} = 0.2344$.

Example 3.9. Convert the given decimal to a fraction then to percent:

a) $0.55 = \dfrac{55}{100}$ which is 55%.

b) $8.09 = \dfrac{809}{100}$ which is 809%.

c) $98.08 = \dfrac{9808}{100}$ which is 9808%.

d) $0.5 = \dfrac{50}{100}$ which is 50%.

Exit Problem

Divide: $782.56 \div 3.2$

Chapter 4

Evaluating Expressions

If a family pays \$30 per phone line, and \$20 per 1GB data, per month to a phone service provider, we can write a mathematical expression to represent the cost that this family pays per month.

Let's use x to represent the number of phone lines that the family has, and y to represent the number of GB of data the family uses. Then $30x + 20y$ is a mathematical expression that represents the cost of the family phone services per month.

$30x + 20y$ is an algebraic expression. A mathematical expression that consists of variables, numbers and algebraic operations is called an **algebraic expression**.

Each algebraic expression can contain several terms. For example, the expression above contains two terms: $30x$ and $20y$. The numerical factor of each term is called a coefficient. The coefficients of the terms above are 30 and 20, respectively. When considering a variable term, we see that it is composed of a **numerical coefficient** and a **variable part**. In the term $30x$ the numerical coefficient is 30 and the variable part is x.

The value of an algebraic expression can vary. For example, the value of the expression above can vary depending on the number of phone lines and the number of GB of data the family uses.

If $x = 2$ (the family uses 2 lines), and $y = 3$ (the family uses 3GB of data), then the cost of the family phone services is $30 \cdot 2 + 20 \cdot 3 = \$120$ for this month.

If $x = 4$, and $y = 2$, then the cost of the family phone devices is $30 \cdot 4 + 20 \cdot 2 = \$160$ for this month.

Finding the value of the expressions when the variables are substituted by given values is called evaluating an algebraic expression.

Example 4.1. The algebraic expression $5x^3y - 2y^2 - z + 4$, which we write using only addition as $5x^3y + (-2y^2) + (-z) + 4$, contains four terms: $5x^3y, -2y^2, -z$

and $+4$. The first three terms are **variable terms** and the 4 is the **constant term**. Notice that the coefficient of $-z$ is -1, and we usually write $-z$ instead of $-1z$. In the same way, if the coefficient is 1, we usually omit it.

This process of finding the value of an algebraic expression for particular values of its variables is called **evaluating an expression**.

Evaluating an expression

1. Replace each variable by the given numerical value.

2. Simplify the resulting expression. Be careful to follow the order of operations.

Helpful tip: When evaluating a variable expression containing a fraction bar, don't forget to work out the numerator and denominator separately (being careful to follow the order of operations as you do so); finally, divide the numerator by the denominator.

Example 4.2. Evaluate if $a = 1$, $b = 2$, $c = 4$, and $d = -1$:

a) $8b$
$$= 8(2)$$
$$= 16$$

b) $a + c$
$$= 1 + 4$$
$$= 5$$

c) $a - d$
$$= 1 - (-1)$$
$$= 1 + 1$$
$$= 2$$

d) $5ab$

$$= 5(1)(2)$$
$$= 10$$

e) $\dfrac{d + a}{b}$

$$= \dfrac{(-1) + 1}{2}$$

$$= \dfrac{0}{2}$$

$$= 0$$

f) $\dfrac{7b + c}{a - d}$

$$= \dfrac{7(2) + 4}{1 - (-1)}$$

$$= \dfrac{14 + 4}{1 + 1}$$

$$= \dfrac{18}{2}$$

$$= 9$$

Example 4.3. Evaluate if $a = -3$, $b = 5$, $c = -2$, and $d = 7$:

a) $4c - 2b$

$$= 4(-2) - 2(5)$$
$$= -8 - 10$$

$$= (-8) + (-10)$$
$$= -18$$

b) $b^2 + b$
$$= 5^2 + 5$$
$$= 25 + 5$$
$$= 30$$

c) $3c^2$
$$= 3(-2)^2$$
$$= 3(4)$$
$$= 12$$

d) $(c + a)(c^2 - ac + a^2)$
$$= ((-2) + (-3)) \left((-2)^2 - (-3)(-2) + (-3)^2\right)$$
$$= (-5)(4 - (-3)(-2) + 9)$$
$$= (-5)(4 - 6 + 9)$$
$$= (-5)(7)$$
$$= -35$$

e) $4b + 5d - \dfrac{c}{a}$

$$= 4(5) + 5(7) - \frac{(-2)}{(-3)}$$

$$= 4(5) + 5(7) - \frac{2}{3}$$

$$= 20 + 35 - \frac{2}{3}$$

$$= \frac{60}{3} + \frac{105}{3} - \frac{2}{3}$$

$$= \frac{60 + 105 - 2}{3}$$

$$= \frac{163}{3}$$

$$= 54\frac{1}{3}$$

f) $-d^2 = -(7)^2 = -49$

Application

Here are some applications that come from geometry.

Example 4.4.

a) Find the perimeter of a rectangle with length 33cm and width 17cm.

$$
\begin{aligned}
P &= 2l + 2w \\
&= 2(33\text{cm}) + 2(17\text{cm}) \\
&= 66\text{cm} + 34\text{cm} \\
&= 100\text{cm}
\end{aligned}
$$

b) Find the area of a rectangle that is 12cm long and 16cm wide.

$$
\begin{aligned}
A &= l \cdot w \\
&= 12\text{cm} \cdot 16\text{cm} \\
&= 192\text{cm}^2
\end{aligned}
$$

c) Find the area of a triangle with height of 20in and a base of 30in.

$$
\begin{aligned}
A &= \frac{1}{2}b \cdot h \\
&= \frac{1}{2}20\text{in} \cdot 30\text{in} \\
&= 300\text{in}^2
\end{aligned}
$$

d) Find the area of a circle with radius 7cm, round your answer to the nearest tenth.

$$
\begin{aligned}
A &= \pi r^2 \\
&= \pi(7\text{cm})^2 \\
&\approx 3.14 \cdot 49\text{cm}^2 \\
&= 153.86\text{cm}^2 \\
&\approx 153.9\text{cm}^2
\end{aligned}
$$

Exit Problem

Evaluate if $a = -2 : -3a^2 - 4a - 16$

Chapter 5

Properties of Exponents

Multiplication Properties

Recall from chapter 1 that $5^3 = 5 \cdot 5 \cdot 5$. In the same way we have

$$x^3 = x \cdot x \cdot x$$

Example 5.1. Perform the given operations:

a) $x^3 \cdot x^5 = \underbrace{x \cdot x \cdot x}_{=x^3} \cdot \underbrace{x \cdot x \cdot x \cdot x \cdot x}_{=x^5} = x^8$

b) $(x^3)^4 = x^3 \cdot x^3 \cdot x^3 \cdot x^3 = \underbrace{x \cdot x \cdot x} \cdot \underbrace{x \cdot x \cdot x} \cdot \underbrace{x \cdot x \cdot x} \cdot \underbrace{x \cdot x \cdot x} = x^{12}$

c) $(x \cdot y)^4 = \underbrace{x \cdot y} \cdot \underbrace{x \cdot y} \cdot \underbrace{x \cdot y} \cdot \underbrace{x \cdot y} = x^4 y^4$

We can summarize these examples into the following useful rules:

> **Multiplication and Exponentiation Rules**
> For any integers n and m
>
> 1. $x^n x^m = x^{n+m}$
>
> 2. $(x^n)^m = x^{n \cdot m}$
>
> 3. $(xy)^n = x^n y^n$

Example 5.2. Perform the given operation using the multiplication properties of exponents and write your answer in simplest form:

a) $b^2 \cdot b^3 = b^{2+3} = b^5$ (recall the meaning of exponents)

b) $x^8 x^7 = x^{8+7} = x^{15}$ (note that juxtaposition indicates multiplication)

c) $a^8 a^9 a^{14} = a^{8+9+14} = a^{31}$

d) $4x^4 \cdot 7x^6 = (4 \cdot 7)(x^4 \cdot x^6) = 28x^{10}$

e) $5xy^3 \cdot 6y = (5 \cdot 6)(x)(y^3 \cdot y) = 30xy^4$

f) $(5x^4 y^2)(2x^7 y^3) = 10x^{4+7} y^{2+3} = 10x^{11} y^5$

g) $(2x)^3 = 2^3 x^3 = 8x^3$

h) $(-4a^2 b^5)^3 = (-4)^3 a^{2 \cdot 3} b^{5 \cdot 3} = -64a^6 b^{15}$

i) $(-5r^3 s)^2 \cdot (2r^4 s^3)^3 \cdot (-r^2 s^2) = (-5)^2 r^{3 \cdot 2} s^2 \cdot 2^3 r^{4 \cdot 3} s^{3 \cdot 3} \cdot (-1)r^2 s^2$

$$= 25r^6 s^2 \cdot 8r^{12} s^9 \cdot (-1)r^2 s^2$$

$$= -200\, r^{20} s^{13}$$

Division Properties

Example 5.3. Perform the given operation:

a) $\dfrac{x^5}{x^3} = \dfrac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \dfrac{x \cdot x}{1} = \dfrac{x^2}{1} = x^2$

b) $\dfrac{x^3}{x^5} = \dfrac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x} = \dfrac{1}{x \cdot x} = \dfrac{1}{x^2}$

c) $\left(\dfrac{x}{y}\right)^5 = \dfrac{x}{y} \cdot \dfrac{x}{y} \cdot \dfrac{x}{y} \cdot \dfrac{x}{y} \cdot \dfrac{x}{y} = \dfrac{x \cdot x \cdot x \cdot x \cdot x}{y \cdot y \cdot y \cdot y \cdot y} = \dfrac{x^5}{y^5}$

We can summarize these examples into the following useful rules:

Division Rules

For any integers n and m

1. $\dfrac{x^n}{x^m} = x^{n-m}$

2. $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

Example 5.4. Perform the given operation using the division properties of exponents and state your answer in simplest form:

a) $\dfrac{b^8}{b^7} = b^{8-7} = b^{8+(-7)} = b^1 = b$

b) $\dfrac{x^{12}y^2}{x^8 y} = x^{12-8}y^{2-1} = x^{12+(-8)}y^{2+(-1)} = x^4 y^1 = x^4 y$

c) $\frac{8x^6y^2}{6x^5y^7} = \frac{4x^{6-5}}{3y^{7-2}} = \frac{4x^1}{3y^5} = \frac{4x}{3y^5}$

d) $\frac{(m^2)^3(n^4)^5}{(m^3)^3} = \frac{m^6 n^{20}}{m^9} = \frac{n^{20}}{m^3}$

e) $\left(\frac{3a^2b^4}{9c^3}\right)^2 = \left(\frac{a^2b^4}{3c^3}\right)^2 = \frac{a^{2\cdot2}b^{4\cdot2}}{3^2c^{3\cdot2}} = \frac{a^4b^8}{9c^6}$

Another way to simplify this correctly is this:
$\left(\frac{3a^2b^4}{9c^3}\right)^2 = \frac{3^2a^{2\cdot2}b^{4\cdot2}}{9^2c^{3\cdot2}} = \frac{9a^4b^8}{81c^6} = \frac{a^4b^8}{9c^6}$

Zero Exponent

Recall from chapter 1 that $(-7)^0 = 1$ and $8^0 = 1$. In the same way we have $x^0 = 1$.

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Zero Exponent

$a^0 = 1$

</div>

Example 5.5. Evaluate.

a) $15^0 = 1$

b) $(-15)^0 = 1$

c) $-15^0 = -1$

d) $(15x)^0 = 1$

e) $15x^0 = 15 \cdot 1 = 15$

Negative Exponents

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Negative Exponents
For any integer n

1. $a^{-n} = \dfrac{1}{a^n}$

2. $\dfrac{1}{a^{-n}} = a^n$

3. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

</div>

Note 5.6. The negative exponent rules can be used to switch terms from numerator to denominator or vice versa, and is useful to write expressions using *positive* exponents only.

Example 5.7. Write the given expression using positive exponents only.

a) $\dfrac{x^{-3}y^2}{z^4} = \dfrac{y^2}{x^3 z^4}$

b) $\dfrac{x^4}{y^{-5}z^2} = \dfrac{x^4 y^5}{z^2}$

Example 5.8. Perform the given operation and write your answer using positive exponents only.

a) $\dfrac{x^4y^2}{x^3y^{-3}} = x^{4-3}y^{2-(-3)} = x^{4+(-3)}y^{2+3} = xy^5$

b) $(x^2y) \cdot (xy^{-4}) = x^{2+1}y^{1+(-4)} = x^3y^{-3} = \dfrac{x^3}{y^3}$

Exit Problem

Simplify: $\left(\dfrac{8y}{3x^3}\right)^3$

Chapter 6

Scientific Notation

To write 1 trillion (1 followed by 12 zeros) or 1 googol (1 followed by 100 zeroes) takes a lot of space and time. There is a mathematical scientific notation which is very useful for writing very big and very small numbers.

Example 6.1. Large numbers in scientific notation:

a) 1 trillion is written as 1×10^{12} in scientific notation.

b) 4 trillion is written as 4×10^{12} in scientific notation.

c) 1 googol is written as 1×10^{100} in scientific notation.

Example 6.2. Small numbers in scientific notation:

a) 0.00000547 is written as 5.47×10^{-6} in scientific notation.

b) 0.00031 is written as 3.1×10^{-4} in scientific notation.

The number $45,600,000$ is a large number, and, basically is $4.56 \times 10,000,000$. So, it can be written as 4.56×10^7.

Similarly, if we consider the number 0.00006772. This is a small number which is $6.772 \times \dfrac{1}{100000}$. That is, it can be written as 6.772×10^{-5}.

The numbers 4.56×10^7 and 6.772×10^{-5} are said to be written in *scientific notation* because the number before the power of 10 is greater than (or equal) to 1 and less than 10, and the decimal number is followed by multiplication by a power of 10.

Standard Form and Scientific Notation

Standard form	Scientific notation
$34,500,000,000$	3.45×10^{10}
0.00000000889	8.89×10^{-9}

Recall from chapter 3 how multiplying or dividing a decimal number by $10, 100, 100, \ldots$ affects the position of the decimal point.

Example 6.3. The given numbers are not in scientific notation. Modify each so that your answer is in scientific notation:

a) $1500 = 1.5 \times 10^3$

b) $225000 = 2.25 \times 10^5$

c) $0.0155 = 1.55 \times 10^{-2}$

d) $0.00000094 = 9.4 \times 10^{-7}$

Example 6.4. The given numbers are not in scientific notation (look at the decimal number and see that it is either less than 1 or greater than 10). Modify each so that your answer is in scientific notation:

a) $56.7 \times 10^8 = 5.67 \times 10^9$

b) $88.9 \times 10^{-7} = 8.89 \times 10^{-6}$

c) $0.55 \times 10^9 = 5.5 \times 10^8$

d) $0.88 \times 10^{-4} = 8.8 \times 10^{-5}$

Helpful tip: Note that the given numbers were not in scientific notation because the decimal number was either greater than 10 or less than 1. To modify the decimal and rewrite the given number in scientific notation, we either increase its size, and thus we must decrease the size of the exponent, or, we decrease its size, and thus, we must increase the size of the exponent.

Multiplication and Division using Scientific Notation

By grouping the decimal numbers together, and the power of 10 terms together, it becomes easy to multiply and divide numbers in scientific notation. First, we need to recall the properties of exponents (we only need base 10 for this section):

Properties of Exponents (for base 10)

1. Product Property
$10^m \cdot 10^n = 10^{m+n}$
Examples: $10^2 \cdot 10^5 = 10^{2+5} = 10^7$ and $10^{-9} \cdot 10^3 = 10^{-9+3} = 10^{-6}$

2. Quotient Property
$\dfrac{10^m}{10^n} = 10^{m-n}$
Examples: $\dfrac{10^5}{10^3} = 10^{5-3} = 10^2$ and $\dfrac{10^5}{10^{-4}} = 10^{5-(-4)} = 10^{5+4} = 10^9$

Example 6.5. Perform the given operation:

a) $(2.3 \times 10^8)(3 \times 10^{-4}) = (2.3 \cdot 3) \times (10^8 \cdot 10^{-4}) = (2.3 \cdot 3) \times 10^{8+(-4)} = 6.9 \times 10^4$

b) $\dfrac{6.4 \times 10^{-9}}{3.2 \times 10^{-5}} = \dfrac{6.4}{3.2} \times \dfrac{10^{-9}}{10^{-5}} = \dfrac{6.4}{3.2} \times 10^{-9-(-5)} = 2 \times 10^{-4}$

Helpful tip: Notice how when we multiplied, we added the exponents, and when we divided, we subtracted the exponent in the denominator from the exponent in the numerator. We simply followed the exponent rules.

Example 6.6. Perform the given operation and write your answer in scientific notation:

a) $(6.2 \times 10^8)(3.0 \times 10^7) = 6.2 \cdot 3 \times 10^{8+7} = 18.6 \times 10^{15} = 1.86 \times 10^{16}$

b) $\dfrac{(4 \times 10^5)}{8 \times 10^{-3}} = \dfrac{4}{8} \times 10^{5-(-3)} = 0.5 \times 10^8 = 5.0 \times 10^7$

   Note: Writing 5.0×10^7 is the same as writing 5×10^7. They are interchangeable.

Example 6.7. Perform the given operation and write your answer in scientific notation:

$$\frac{(2.1 \times 10^3)(3.2 \times 10^{-8})}{(2 \times 10^4)(3 \times 10^9)} = \frac{6.72 \times 10^{-5}}{6 \times 10^{13}} = 1.12 \times 10^{-18}$$

Example 6.8. The debt of a nation is 7 trillion dollars, and there are 300 thousand inhabitants. If the debt was distributed evenly among all the inhabitants, how much would each person have to pay to pay off the debt?

To answer this question we have to divide: $\dfrac{7 \text{ trillion dollars}}{300 \text{ million people}}$.

Since we are dealing with large numbers, we change each to scientific notation and perform the division to find the amount to be paid per person:

$\dfrac{7 \times 10^{15}}{3 \times 10^{11}}$ dollars per person $= \dfrac{7}{3} \times 10^4$ dollars per person $\approx 2.3333 \times 10^4$ dollars per person.

In standard form, the amount is $\$23,333$ per person.

Exit Problem

Compute and write the answer in scientific notation:

$$\frac{(6.2 \times 10^2)(1.5 \times 10^{-4})}{3.1 \times 10^{-9}}$$

Chapter 7

Polynomials

A **polynomial** is a sum of monomials. So, expressions like:

$$x^2 + 3x + 7$$

$$-2x^3 + 4x^2 - 5x + 2$$

$$x + x^2$$

$$-4x^3$$

$$x^2 y + \frac{xyz^2}{6} - 8y^2 z^2$$

are examples of polynomials. However, the following are *not* polynomials:

$$\frac{x^2 + 3x + 4}{x + 5}$$

$$\frac{2x^2 yz^3}{-xy^2}$$

$$-4x\sqrt{6x}$$

The **degree** of a polynomial is the highest power of the variable(s) that has a non-zero coefficient.

Example 7.1. The degree of $-2x^3 + 4x^2 - 5x + 2$ is 3.

Example 7.2. Determine whether the given expression is a polynomial and if so, find its degree.

a) $5x^3 + 4x^2 - 3x + 7$

This is a polynomial of degree 3.

b) $\dfrac{2}{x^2+3x-4}$

This is a not a polynomial.

c) $2x^3 + 5x^4 + 3x - 8$

This is a polynomial of degree 4.

d) $6.2 \times 10^{-5} x^8$

This is a polynomial of degree 8.

A polynomial with one term is called a **monomial**. For example, $2a^5$ and $-3xy^2$ are monomials. A polynomial with two terms is called a **binomial**. $5x^2 + 3x$ is an example of a binomial. A polynomial with three terms is called a **trinomial**. $3x^2 + 5x - 1$ is a trinomial. It has three terms: $3x^2$, $5x$ and -1.

Just as we did in chapter 4 when evaluating expressions, we can evaluate polynomials as well.

Example 7.3. Evaluate the given polynomial at the given value of the variable(s):

a) $3x + 7$ when $x = 2$:

$3(2) + 7 = 6 + 7 = 13$

b) $x^2 + 3x + 2$ when $x = 5$:

$(5)^2 + 3(5) + 2 = 25 + 15 + 2 = 42$

c) $2x^3 + 4x^2 - 3x$ when $x = -2$:
$2(-2)^3 + 4(-2)^2 - 3(-2) = 2(-8) + 4(4) - (-6) = -16 + 16 + 6 = 6$

d) $-4x^7 - 3x^4$ when $x = -1$:
$-4(-1)^7 - 3(-1)^4 = -4(-1) - 3(1) = 4 - 3 = 1$

e) $2x^2y - 5x^3y^2$ when $x = -3$ and $y = 2$:
$2(-3)^2(2) - 5(-3)^3(2)^2 = 2(9)(2) - 5(-27)(4)$

$$= 18(2) + 135(4)$$

$$= 36 + 540$$

$$= 576$$

Function Notation

A specific kind of notation, called function notation, can be used to represent polynomials. This notation uses a letter (the name of the function) and the variable at hand (for example, x.)

Example 7.4. Function notation:

a) $f(x) = 3x + 7$ is representing the polynomial $3x + 7$ as a function called f. The x in $f(x)$ is to indicate that the variable in the polynomial is x.

b) $g(x) = x^2 + 3x + 2$ is representing the polynomial $x^2 + 3x + 2$ as a function called g. Again, the x in $g(x)$ is to indicate that the variable in the polynomial is x.

c) $f(x, y) = 2x^2y - 5x^3y^2$ is representing the polynomial $2x^2y - 5x^3y^2$ as a function called f. The x and y in $f(x, y)$ are to indicate that the variables in the polynomial are x and y.

You will learn about functions and function notation in a pre-calculus class, but, here we use the notation because it facilitates asking to evaluate a polynomial at a given value of the variable(s), as we saw in Example 7.3.

So, for example, find $f(2)$ when $f(x) = 3x + 7$ is asking to evaluate the polynomial $3x + 7$ when $x = 2$. So, $f(2) = 3 \cdot 2 + 7 = 13$.

Example 7.5. $f(x) = x^2 - 1$. Find $f(-3)$.
$f(-3) = (-3)^2 - 1 = 9 - 1 = 8$

Example 7.6. $g(x) = -3x^3 + 5$. Find $g(-2)$.
$g(-2) = -3 \cdot (-2)^3 + 5 = -3 \cdot (-8) + 5 = 24 + 5 = 29$

> **Exit Problem**
> Evaluate $f(-1)$ for the function $f(x) = -2x^3 + 3x^2 - x$

Chapter 8

Adding and Subtracting Polynomial Expressions

Polynomials can be **added**, **subtracted**, **multiplied** and **divided**. When **adding** or **subtracting**, we can only combine terms that are **like terms**.

What Are Like Terms?

Consider the expression

$$5x^4y^2 + 6x^3y - 7y^2x^4.$$

First, we rewrite the given expression using addition only:

$$5x^4y^2 + 6x^3y + (-7y^2x^4)$$

Here the terms are: $5x^4y^2$, $6x^3y$, and $-7y^2x^4$. The terms are added to get the expression.

Two terms are either "unlike" or "like".

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Like Terms

Like terms are terms that have the same exponents on the same variables.

</div>

For example, in the above expression $5x^4y^2$ and $-7y^2x^4$ are like terms since y has the same exponent (2) and the x has the same exponent (4). On the other hand, $5x^4y^2$ and $6x^3y^2$ are unlike terms since x^4 appears in the first term but x^3 appears in the second.

Example 8.1. Consider the expression:

$$-2x^5y^2 - 5x^4y^2 + 6x^5y^2$$

The like terms are: $-2x^5y^2$ and $6x^5y^2$.

Adding or Subtracting Like Terms

> **Adding or Subtracting Like Terms**
> We can only add or subtract like terms, and, we do so by
> adding or subtracting their coefficients.

Example 8.2. $-2x^5y^2 + 6x^5y^2 = 4x^5y^2$ so that our expression in the previous example can be simplified:

$$-2x^5y^2 - 5x^4y^2 + 6x^5y^2 = 4x^5y^2 - 5x^4y^2.$$

Now, let's add and subtract polynomials.

Example 8.3. Add or subtract the polynomials.

a) $(3x^2 + 5x + 6) + (4x^2 + 3x - 8) = 3x^2 + 5x + 6 + 4x^2 + 3x - 8$

$$= 7x^2 + 8x - 2$$

b) $(x^4 - 2x^3 + 6x) + (5x^3 + 2x^2 + 9x + 4)$

$$= x^4 - 2x^3 + 6x + 5x^3 + 2x^2 + 9x + 4$$

$$= x^4 + 3x^3 + 2x^2 + 15x + 4$$

c) $(2ab^2 - 3a^2 - 7ab) + (-a^2 - 5a^2b)$

$$= 2ab^2 - 3a^2 - 7ab - a^2 - 5a^2b$$

$$= 2ab^2 - 4a^2 - 7ab - 5a^2b$$

Note: Above, the only like terms that can be combined are $-3a^2$ and $-a^2$. The remaining terms cannot be combined any further.

d) $(3x^2 + 5x + 6) - (4x^2 + 3x - 8) = 3x^2 + 5x + 6 - 4x^2 - 3x + 8$

$$= -x^2 + 2x + 14$$

Note: Removing the parenthesis when subtracting changes the sign for all the terms of the polynomial that is being subtracted.

e) $(5m + 3n) - (7m + 2n) = 5m + 3n - 7m - 2n$

$$= -2m + n$$

f) Subtract $4a^2 - 5a$ from $6a + 4$.
$(6a + 4) - (4a^2 - 5a) = 6a + 4 - 4a^2 + 5a$

$$= -4a^2 + 11a + 4$$

Note: Subtracting $4a^2 - 5a$ from $6a + 4$ requires to write $6a + 4$ first, and then to subtract $4a^2 - 5a$ from it. Reversing the order would yield a wrong answer!

g) Subtract $-3q + 4pq - 2p$ from $-9p$.
$-9p - (-3q + 4pq - 2p) = -9p + 3q - 4pq + 2p$

$$= -7p + 3q - 4pq$$

Exit Problem

Simplify: $(9m^2n - 15mn^2) - (3mn^2 + 2m^2n)$

Chapter 9

Multiplying Polynomial Expressions

In this chapter, we multiply polynomials.

For the **multiplication** of a monomial by a polynomial, we need to distribute the monomial to multiply each term of the polynomial.

Distributive Law

$$a \quad (\quad b \quad + \quad c \quad) = \quad a\,b + a\,c$$

By distributing factors, we can multiply a monomial by a polynomial.

Example 9.1. Multiply the terms and simplify.

a) $3\left(4x^2 + 5x\right) = 12x^2 + 15x$

b) $2p\left(4p - 7q\right) = 8p^2 - 14pq$

c) $-3a\left(a^2 - 4a + 5\right) = -3a^3 + 12a^2 - 15a$

d) $6x + 7y + 4\,(3x + 2y) = 6x + 7y + 12x + 8y$

$$= 18x + 15y$$

e) $2\,(3x^2 + 5) + 3x\,(4x - 2) = 6x^2 + 10 + 12x^2 - 6x$

$$= 18x^2 - 6x + 10$$

f) $y^2\,(y + 3) - 2y\,(-y + 5) - 4\,(2y^2 + 12y + 1)$

$$= y^3 + 3y^2 + 2y^2 - 10y - 8y^2 - 48y - 4$$

$$= y^3 - 3y^2 - 58y - 4$$

g) $-4ab\,(-a - 3b) + 2a\,(8b^2 - 7ab)$

$$= 4a^2b + 12ab^2 + 16ab^2 - 14a^2b$$

$$= -10a^2b + 28ab^2$$

When multiplying general polynomials, we need to have a more general distributive law. We show how this is done for the product of two binomials.

Rule 9.2. To multiply binomials, we need to multiply each term of the first polynomial with each term of the second polynomial. This procedure is known under the name **<u>FOIL</u>**, which stands for **First**, **Outer**, **Inner**, **Last**.

For example, when multiplying $(x + 3)$ with $(x + 5)$, we multiply the terms

$$\left(\quad x \quad + \quad 3 \quad \right) \left(\quad x \quad + \quad 5 \quad \right)$$

First	x^2
Outer	$+5x$
Inner	$+3x$
Last	$+15$

The products of the first terms $(x \cdot x = x^2)$, the outer terms $(x \cdot 5 = 5x)$, the inner terms $(3 \cdot x = 3x)$, and the last terms $(3 \cdot 5 = 15)$ are added to give the result. Therefore, applying FOIL and combining like terms, we obtain:

$$(x+3)(x+5) \quad = \quad x^2 + 5x + 3x + 15 \quad = \quad x^2 + 8x + 15.$$

Example 9.3. Multiply and simplify.

a) $(x+2)(x-7) = x^2 - 7x + 2x - 14$

$$= x^2 - 5x - 14$$

b) $(2a+3b)(4a+5b) = 8a^2 + 10ab + 12ab + 15b^2$

$$= 8a^2 + 22ab + 15b^2$$

c) $(m-4n)^2 = (m-4n)(m-4n)$

$$= m^2 - 4mn - 4mn + 16n^2$$

$$= m^2 - 8mn + 16n^2$$

d) $-5p\,(3p+2q)^2 = -5p\,(3p+2q)(3p+2q)$

$$= -5p\,(9p^2 + 6pq + 6pq + 4q^2)$$

$$= -5p\,(9p^2 + 12pq + 4q^2)$$

$$= -45p^3 - 60p^2q - 20pq^2$$

Above we have chosen to first evaluate the square $(3p+2q)(3p+2q)$. Alternatively, we could also have first multiplied $-5p\,(3p+2q)$, which also gives the correct result:

$$-5p\,(3p+2q)^2 = -5p\,(3p+2q)(3p+2q)$$

$$= (-15p^2 - 10pq)(3p+2q)$$

$$= -45p^3 - 30p^2q - 30p^2q - 20pq^2$$

$$= -45p^3 - 60p^2q - 20pq^2$$

e) $(r+s)^2 - (r+s)(r-s) = (r+s)(r+s) - (r+s)(r-s)$

$$= (r^2 + rs + rs + s^2) - (r^2 - rs + rs - s^2)$$

$$= (r^2 + 2rs + s^2) - (r^2 - s^2)$$

$$= r^2 + 2rs + s^2 - r^2 + s^2$$

$$= 2rs + 2s^2$$

f) $2xy - 3\,(5x+y)(5x-y) + 4y\,(3x+2y)$

$$= 2xy - 3\,(25x^2 - 5xy + 5xy - y^2) + 12xy + 8y^2$$

$$= 2xy - 3\,(25x^2 - y^2) + 12xy + 8y^2$$

$$= 2xy - 75x^2 + 3y^2 + 12xy + 8y^2$$

$$= -75x^2 + 14xy + 11y^2$$

g) $(2x - 1)(x^2 - 4x + 6) = (2x + (-1))(x^2 - 4x + 6)$

$$= 2x(x^2 - 4x + 6) + (-1)(x^2 - 4x + 6)$$

$$= 2x \cdot x^2 + 2x \cdot (-4x) + 2x \cdot 6 + (-1) \cdot x^2$$

$$+ (-1) \cdot (-4x) + (-1) \cdot 6$$

$$= 2x^3 - 8x^2 + 12x - x^2 + 4x - 6$$

$$= 2x^3 - 9x^2 + 16x - 6$$

Exit Problem

Multiply: $(5x - 2)(x^2 - 3x - 7)$

Simplify: $-3a^3b^2 + 4a^2(a + 2ab^2) - 7a^3$

Chapter 10

Dividing Polynomials

In the previous chapter, we added, subtracted, and multiplied polynomials. Now, what remains is dividing polynomials. We will only consider division of a polynomial by a monomial. The division of a monomial by a monomial was already considered in chapter 5, which we now recall.

Rule 10.1. We recall the rules for dividing variables.

$$\frac{x^n}{x^m} = x^{n-m} \qquad \text{for any integers } n \text{ and } m$$

Example 10.2. Simplify.

a) $\dfrac{27x^3y^5}{3x^2y^3} = \dfrac{27x^{3-2}y^{5-3}}{3} = 9x^1y^2 = 9xy^2$

b) $\dfrac{-56a^8b^6c^4}{-7a^5b\,c^4} = \dfrac{-56\,a^{8-5}b^{6-1}c^{4-4}}{-7} = 8a^3b^5c^0 = 8a^3b^5$

Of course, when the power of a variable is higher in the denominator than in the numerator, then those variables will remain in the denominator, just as we did in chapter 5.

Example 10.3. Simplify.

a) $\dfrac{42p^7q^4}{-3p^3q^2} = \dfrac{42}{-3}q^{4-2}p^{7-4} = \dfrac{14}{-1}q^2p^3 = -14q^2p^3$

b) $\dfrac{24r^4s^9t^5}{20rs^6t^2} = \dfrac{24}{20}r^{4-1}s^{9-6}t^{5-2} = \dfrac{6}{5}r^3s^3t^3$

We now study how a polynomial can be divided by a monomial. Recall the usual rule for adding fractions with common denominator.

> **Same Denominator Fractions**
> Fractions with common denominator can be added (or subtracted) by adding (or subtracting) the numerators:
>
> Add: $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a+b}{c}$
>
> Subtract: $\dfrac{a}{c} - \dfrac{b}{c} = \dfrac{a-b}{c}$

Reversing the above rule helps us to divide a polynomial by a monomial.

Example 10.4. Simplify as much as possible.

a) $\dfrac{6x+15}{3} = \dfrac{6x}{3} + \dfrac{15}{3} = 2x + 5$

b) $\dfrac{14x^3-8x^2}{2x} = \dfrac{14x^3}{2x} - \dfrac{8x^2}{2x} = 7x^2 - 4x$

c) $\dfrac{14y^6-28y^5+21y^3}{-7y^2} = \dfrac{14y^6}{-7y^2} - \dfrac{28y^5}{-7y^2} + \dfrac{21y^3}{-7y^2}$

$$= -2y^4 - (-4y^3) - 3y$$

$$= -2y^4 + 4y^3 - 3y$$

d) $\dfrac{a^2b^4-4ab^3-2a^4b^2}{ab^2} = \dfrac{a^2b^4}{ab^2} - \dfrac{4ab^3}{ab^2} - \dfrac{2a^4b^2}{ab^2}$

$$= ab^2 - 4b - 2a^3$$

e) $\dfrac{-6r^5t^4+30r^4s^2t^5-42r^3s^2t^3}{-6rt^3} = \dfrac{-6r^5t^4}{-6rt^3} + \dfrac{30r^4s^2t^5}{-6rt^3} - \dfrac{42r^3s^2t^3}{-6rt^3}$

$$= r^4t - 5r^3s^2t^2 - (-7r^2s^2)$$

$$= r^4t - 5r^3s^2t^2 + 7r^2s^2$$

Exit Problem

Simplify: $\dfrac{27x^2y - 3xy + 15xy^2}{-3xy}$

Chapter 11

Simplifying Square Roots

Finding a square root of a number is the inverse operation of squaring that number. Remember, the square of a number is that number times itself. For example, $5^2 = 5 \cdot 5 = 25$ and $(-5)^2 = (-5) \cdot (-5) = 25$.

The square root of a number n, written as $\sqrt{n}$, is the **positive number** that gives n when multiplied by itself. For example, $\sqrt{25} = 5$ and not -5 because 5 is the **positive** number that multiplied by itself gives 25.

The perfect squares are the squares of whole numbers: $1 = 1^2, 4 = 2^2, 9 = 3^2, 16 = 4^2, 25 = 5^2, 36 = 6^2, 49 = 7^2, 64 = 8^2, 81 = 9^2, 100 = 10^2, \ldots$ and finding their square roots is straightforward. So $\sqrt{16} = \sqrt{4^2} = 4$, $\sqrt{100} = \sqrt{10^2} = 10$.

What about $\sqrt{50}$? Can you think of a number you multiply by itself and the answer is 50? The only thing we can do is simplify the square root. We say that a square root is **simplified** if it has **no perfect square factors**.

So, to simplify $\sqrt{50}$ we first write 50 into its factor and look for perfect squares. $50 = 25 \cdot 2 = 5^2 \cdot 2$. Then, $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{5^2} \cdot \sqrt{2} = 5 \cdot \sqrt{2}$. The justification for separating $\sqrt{5^2}$ and $\sqrt{2}$ is the fact that the square root of a **product** is equal to the product of the square root of each factor:

> **Product Rule for Radicals**
>
> $$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

Example 11.1. Simplify each of the following radical expressions:

a) $\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{2^2 \cdot 6}$

$$= \sqrt{2^2} \cdot \sqrt{6} = 2 \cdot \sqrt{6}$$

b) $\sqrt{108} = \sqrt{36 \cdot 3} = \sqrt{6^2 \cdot 3} = \sqrt{6^2} \cdot \sqrt{3} = 6 \cdot \sqrt{3}$

c) $2 \cdot \sqrt{80} = 2 \cdot \sqrt{16 \cdot 5} = 2 \cdot \sqrt{4^2 \cdot 5}$

$$= 2 \cdot \sqrt{4^2} \cdot \sqrt{5} = 2 \cdot 4 \cdot \sqrt{5}$$

$$= 8 \cdot \sqrt{5}$$

How to combine "like" square roots

We can combine "like" square roots the same way we combined "like terms" in chapter **??**.

> **Like Square Roots**
> Two (or more) square roots are "like" if they have the same quantity under the root.
> **Note:** Always simplify the square root if possible before identifying "like" roots.

Example 11.2. Like square roots:

a) $\sqrt{3}$ and $-6\sqrt{3}$

b) $2\sqrt{5}$ and $-4\sqrt{5}$

c) $\sqrt{7}$ and $\sqrt{28}$, because $\sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4} \cdot \sqrt{7} = 2\sqrt{7}$

d) $\sqrt{90}$ and $\sqrt{250}$ because $\sqrt{90} = \sqrt{9 \cdot 10} = \sqrt{9} \cdot \sqrt{10} = 3\sqrt{10}$ and $\sqrt{250} = \sqrt{25 \cdot 10} = \sqrt{25} \cdot \sqrt{10} = 5\sqrt{10}$

To add or to subtract radicals, we need to first simplify radicals, then combine like radicals.

Example 11.3. Add or Subtract radicals:

a) $\sqrt{160} + \sqrt{490} = \sqrt{16 \cdot 10} + \sqrt{49 \cdot 10} = \sqrt{16} \cdot \sqrt{10} + \sqrt{49} \cdot \sqrt{10} = 4 \cdot \sqrt{10} + 7 \cdot \sqrt{10} = (4 + 7)\sqrt{10} = 11\sqrt{10}$

b) $2\sqrt{27} - 5\sqrt{3} = 2\sqrt{9 \cdot 3} - 5\sqrt{3} = 2\sqrt{9} \cdot \sqrt{3} - 5\sqrt{3} = 2 \cdot 3\sqrt{3} - 5\sqrt{3} = 6\sqrt{3} - 5\sqrt{3} = (6 - 5)\sqrt{3} = 1\sqrt{3} = \sqrt{3}$

c) $4\sqrt{18} - 7\sqrt{8} - 3\sqrt{1} = 4\sqrt{9}\sqrt{2} - 7\sqrt{4}\sqrt{2} - 3 \cdot 1 = 4 \cdot 3\sqrt{2} - 7 \cdot 2\sqrt{2} - 3 = 12\sqrt{2} - 14\sqrt{2} - 3 = (12 - 14)\sqrt{2} - 3 = -2\sqrt{2} - 3$

Note: Only "like" roots can be combined.

How do we simplify non-numerical radicals?

Similar to numbers, variables inside the square root that are squared (raised to the power 2) can be simplified. So, $\sqrt{x^2} = x$ in the same way that $\sqrt{8^2} = 8$. So, we need to find as many multiples of variables that are squared:

$$\sqrt{x^8} = \sqrt{x^2 \cdot x^2 \cdot x^2 \cdot x^2} = \sqrt{x^2} \cdot \sqrt{x^2} \cdot \sqrt{x^2} \cdot \sqrt{x^2} = x \cdot x \cdot x \cdot x = x^4$$

$$\sqrt{x^5} = \sqrt{x^2 \cdot x^2 \cdot x} = \sqrt{x^2} \cdot \sqrt{x^2} \cdot \sqrt{x} = x \cdot x \cdot \sqrt{x} = x^2\sqrt{x}$$

If you have a number and a variable inside the square root (or more than one variable), you work with each one separately. For example:

$$\sqrt{50x^8} = \sqrt{50} \cdot \sqrt{x^8} = 5\sqrt{2} \cdot x^4 = 5x^4\sqrt{2}$$

Example 11.4. Simplify each of the following radical expressions:

a) $\sqrt{y^4x^8} = \sqrt{y^4} \cdot \sqrt{x^8} = \sqrt{y^2 \cdot y^2} \cdot \sqrt{x^8}$

$$= \sqrt{y^2} \cdot \sqrt{y^2} \cdot \sqrt{x^8} = y \cdot y \cdot \sqrt{x^8} = y^2 \cdot \sqrt{x^8}$$

$$= y^2 \cdot x^4$$

$$= y^2x^4$$

b) $\sqrt{200 \cdot m^4} = \sqrt{200} \cdot \sqrt{m^4} = \sqrt{100 \cdot 2} \cdot \sqrt{m^4}$

$$= \sqrt{10^2 \cdot 2} \cdot \sqrt{m^4} = \sqrt{10^2} \cdot \sqrt{2} \cdot \sqrt{m^4} = 10 \cdot \sqrt{2} \cdot m^2$$

$$= 10 \cdot m^2 \cdot \sqrt{2} = 10m^2\sqrt{2}$$

c) $m^3 \cdot \sqrt{200 \cdot m^4} = m^3 \cdot \sqrt{200} \cdot \sqrt{m^4} = m^3 \cdot \sqrt{100 \cdot 2} \cdot \sqrt{m^4}$

$$= m^3 \cdot \sqrt{10^2 \cdot 2} \cdot \sqrt{m^4} = m^3 \cdot \sqrt{10^2} \cdot \sqrt{2} \cdot \sqrt{m^4}$$

$$= m^3 \cdot 10 \cdot \sqrt{2} \cdot m^2 = 10m^5 \cdot \sqrt{2}$$

$$= 10m^5\sqrt{2}$$

d) $2\sqrt{63x^3} = 2 \cdot \sqrt{63} \cdot \sqrt{x^3} = 2 \cdot \sqrt{9 \cdot 7} \cdot \sqrt{x^3}$

$$= 2 \cdot \sqrt{3^2 \cdot 7} \cdot \sqrt{x^2 \cdot x} = 2 \cdot \sqrt{3^2} \cdot \sqrt{7} \cdot \sqrt{x^2} \cdot \sqrt{x}$$

$$= 2 \cdot 3\sqrt{7} \cdot x \cdot \sqrt{x} = 6x \cdot \sqrt{7x} = 6x\sqrt{7x}$$

Similar to the product rule, the quotient rule allows us to separate square roots as follows:

> **Quotient Rule for Radicals**
>
> $$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

And notice that both rules can be read from right to left as follows:

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

This provides us with the necessary tools to combine and simplify square roots, when the operation is **multiplication** or **division** only.

Example 11.5. Simplify completely:

a) $\dfrac{\sqrt{15}\sqrt{70}}{\sqrt{5}} = \dfrac{\sqrt{15 \cdot 70}}{\sqrt{5}}$

$$= \sqrt{\frac{15 \cdot 70}{5}} = \sqrt{\frac{5 \cdot 3 \cdot 7 \cdot 5 \cdot 2}{5}}$$

$$= \sqrt{5 \cdot 3 \cdot 2 \cdot 7} = \sqrt{210}$$

b) $-x\sqrt{12y^3} \cdot 3y^2\sqrt{15x} = -3xy^2 \cdot \sqrt{12y^3 \cdot 15x} = -3xy^2 \cdot \sqrt{12 \cdot 15xy^3}$

$$= -3xy^2 \cdot \sqrt{4 \cdot 3 \cdot 3 \cdot 5xy^3}$$

$$= -3xy^2 \cdot \sqrt{4 \cdot 3^2 \cdot 5xy \cdot y^2}$$

$$= -3xy^2 \cdot \sqrt{4} \cdot \sqrt{3^2} \cdot \sqrt{5xy} \cdot \sqrt{y^2}$$

$$= -3xy^2 \cdot 2 \cdot 3 \cdot \sqrt{5xy} \cdot y$$

$$= -18xy^3 \cdot \sqrt{5xy}$$

Exit Problem

Simplify: $5\sqrt{24} - 2\sqrt{54} - 3\sqrt{16}$

Chapter 12

Factoring a Monomial from a Polynomial and GCF

We have learned how to multiply polynomials. For example using the distributive property on $2x(x-1)$ we get $2x^2 - 2x$ or using FOIL on $(x+1)(2x-5)$ results in $2x^2 - 3x - 5$. Factoring is an inverse procedure of multiplying polynomials. If we write $2x^2 - 2x = 2x(x-1)$ or $2x^2 - 3x - 5 = (x+1)(2x-5)$, this is called factoring. In general, factoring is a procedure to write a polynomial as a product of two or more polynomials. In the first example $2x$ and $(x-1)$ are called factors of $2x^2 - 2x$. In the second example $(x+1)$ and $(2x-5)$ are factors of $2x^2 - 3x - 5$.

Factoring is extremely useful when we try to solve polynomial equations and simplify algebraic fractions. In the following three chapters, we will learn several methods of factoring.

Factoring the Greatest Common Factor (GCF)

In this section we will explain how to factor the greatest common factor (GCF) from a polynomial. The GCF is the largest polynomial that divides every term of the polynomial. For example, the polynomial $9x^3 + 6x^2 + 12x^4$ is the addition of three terms $9x^3$, $6x^2$ and $12x^4$. From factorization, we have:

$$9x^3 = 3 \cdot 3 \cdot x \cdot x \cdot x = (3x^2)(3x)$$

$$6x^2 = 2 \cdot 3 \cdot x \cdot x = (3x^2)(2)$$

$$12x^4 = 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot x \cdot x = (3x^2)(4x^2).$$

Therefore, the monomial $3x^2$ is the GCF of $9x^3$, $6x^2$ and $12x^4$. Once the GCF is identified, we can use the distributive property to factor out the GCF as follows:

$$9x^3 + 6x^2 + 12x^4 = 3x^2(3x) + 3x^2(2) + 3x^2(4x^2)$$

$$= 3x^2(3x + 2 + 4x^2).$$

Example 12.1. Factor out the GCF from the given polynomial.

a) $6a^4 - 8a$
   The GCF is $2a$.

$$6a^4 - 8a = 2a(3a^3) - 2a(4) = 2a(3a^3 - 4).$$

b) $15ab^3c - 10a^2bc + 25b^2c^3$
   The GCF is $5bc$.

$$15ab^3c - 10a^2bc + 25b^2c^3 = 5bc(3ab^2) - 5bc(2a^2) + 5bc(5bc^2)$$

$$= 5bc(3ab^2 - 2a^2 + 5bc^2).$$

c) $3x^2 - 6x + 12$
   The GCF is 3.

$$3x^2 - 6x + 12 = 3(x^2) - 3(2x) - 3(4) = 3(x^2 - 2x - 4).$$

In the problems we have done so far, the first term of the polynomial has been positive. If the first term of the polynomial is negative, it is better to factor the opposite of the GCF. The procedure is exactly the same, but we need to pay attention to the sign of each term.

Example 12.2. Factor out the GCF from the given polynomial:

a) $-8x^2y + 12xy^2 - 16xy$

   The GCF is $4xy$. Since the first term of the polynomial is negative, we factor out $-4xy$ instead.

$$-8x^2y + 12xy^2 - 16xy = (-4xy)(2x) + (-4xy)(-3y) + (-4xy)(4)$$

$$= -4xy(2x - 3y + 4).$$

Note: Inside the parentheses, the sign has been changed for every term in the remaining factor.

b) $-9b^5 - 15b^4 + 21b^3 + 27b^2$

   Since the first term is negative, we factor out the opposite of the GCF: $-3b^2$.

$$-9b^5 - 15b^4 + 21b^3 + 27b^2$$

$$= (-3b^2)(3b^3) + (-3b^2)(5b^2) + (-3b^2)(-7b) + (-3b^2)(-9)$$

$$= -3b^2(3b^3 + 5b^2 - 7b - 9).$$

Again, notice how inside the parentheses, the sign of each of the remaining terms has been changed.

Factoring Out the Greatest Common Factor (GCF)

Step 1. Identify the GCF of each term of the polynomial.

Step 2. Write each term of the polynomial as a product of the GCF and remaining factor. If the first term of the polynomial is negative, we use the opposite of the GCF as the common factor.

Step 3. Use the distributive property to factor out the GCF.

Factoring by Grouping

In many expressions, the greatest common factor may be a binomial. For example, in the expression $x^2(2x+1)+5(2x+1)$, the binomial $(2x+1)$ is a common factor of both terms. So we can factor out this binomial factor as following:

$$x^2(2x+1) + 5(2x+1) = (2x+1)(x^2+5).$$

This idea is very useful in factoring a four-term polynomial using the grouping method. For example, $6a^2 - 10a + 3ab - 5b$ is a polynomial with four terms. The GCF of all four terms is 1. However, if we group the first two terms and the last two terms, we get $(6a^2 - 10a) + (3ab - 5b)$. Notice that in the first group, $2a$ is a common factor and in the second group, b is a common factor, so we can factor them out: $2a(3a - 5) + b(3a - 5)$. The two terms now have a common binomial factor $3a - 5$. After factoring it out, we get: $2a(3a - 5) + b(3a - 5) = (3a - 5)(2a + b)$. Factoring by grouping is a very useful method when we factor certain trinomials in a later session.

Example 12.3. Factor the given polynomials by the grouping method.

a) $x^3 + 3x^2 + 2x + 6$

$$x^3 + 3x^2 + 2x + 6 = (x^3 + 3x^2) + (2x + 6)$$

$$= x^2(x + 3) + 2(x + 3)$$

$$= (x + 3)(x^2 + 2).$$

b) $6x^2 - 3x - 2xy + y$

$$6x^2 - 3x - 2xy + y = (6x^2 - 3x) + (-2xy + y)$$

$$= 3x(2x - 1) + (-y)(2x - 1)$$

$$= (2x - 1)(3x - y).$$

c) $10ax + 15ay - 8bx - 12by$

$$10ax + 15ay - 8bx - 12by = (10ax + 15ay) + (-8bx - 12by)$$

$$= 5a(2x + 3y) + (-4b)(2x + 3y)$$

$$= (2x + 3y)(5a - 4b).$$

Note: In the last two problems of the example, the third term in the polynomial is negative. When grouping, we need to add an extra addition sign between two groups, and in factoring the second group, factoring the opposite of the GCF is helpful.

Exit Problem

Factor completely: $12a^5b^4c^5 - 36a^6b^3c - 24ab^2$

Factor by grouping: $35xy + 21ty - 15xz - 9tz$

Chapter 13

Factoring the Difference of Two Squares

In this chapter, we will learn how to factor a binomial that is a difference of two perfect squares. We have learned in multiplying polynomials that a product of two conjugates yields a difference of two perfect squares:

$$(a + b)(a - b) = a^2 - \cancel{ab} + \cancel{ab} - b^2 = a^2 - b^2.$$

This indicates that the factor form of $a^2 - b^2$ is $(a + b)(a - b)$, a product of two conjugates. Let's put this as a formula:

> **Factoring the Difference of Two Squares**
>
> $$a^2 - b^2 = (a + b)(a - b)$$

Example 13.1. Factor a difference of two squares.

a) $49 - y^2 = (7)^2 - y^2 = (7 + y)(7 - y)$

b) $16w^2 - x^2y^2 = (4w)^2 - (xy)^2 = (4w + xy)(4w - xy)$

c) $9a^6 - b^4 = (3a^3)^2 - (b^2)^2 = (3a^3 + b^2)(3a^3 - b^2)$

Sometimes, the binomial is not a difference of two perfect squares, but after we factor out the GCF, the resulting binomial is a difference of two perfect squares. Then we can still use this formula to continue factoring the resulting binomial.

Example 13.2. Factor the binomial completely.

a) $18x^3 - 8xy^2 = 2x(9x^2 - 4y^2) = 2x[(3x)^2 - (2y)^2] = 2x(3x + 2y)(3x - 2y)$

b) $3a^5 - 27ab^2 = 3a(a^4 - 9b^2) = 3a[(a^2)^2 - (3b)^2] = 3a(a^2 + 3b)(a^2 - 3b)$

Exit Problem

Factor completely: $16x^2 - 36$

Chapter 14

Factoring Trinomials and Mixed Factoring

Factoring Trinomials $ax^2 + bx + c$ by the ac-Method

We know that multiplying two binomials by the FOIL method results in a four-term polynomial and in many cases it can be combined into a three-term polynomial. For example: $(x+3)(2x+1) = 2x^2+1x+6x+3 = 2x^2+7x+3$. This indicates that if we want to factor the expression $2x^2+7x+3$, we will get a product of two binomials $(x+3)$ and $(2x+1)$, that is, $2x^2+7x+3 = (x+3)(2x+1)$. In this section, we will learn how to reverse the procedure of FOIL to factor trinomials of the form $ax^2 + bx + c$. The procedure is called the ac-Method.

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ac-Method to factor $ax^2 + bx + c; a \neq 0$

Step 1. Find the product ac, that is the product of the coefficients of the first and last terms.

Step 2. Find two integers whose product is ac and whose sum is b. If such an integer pair cannot be found, then the polynomial cannot be factored out.

Step 3. Use the two integers found in step 2 to rewrite the term bx as a sum of two terms.

Step 4. Factor by the grouping method.

For example: Factor $2x^2 + 7x + 3$.

Step 1. The product of ac is $2 \cdot 3 = 6$.

Step 2. We look for two numbers whose product is 6 and whose sum is 7. We can do this by inspection or by writing all pairs of numbers whose product is 6 and calculate the sum for each pair: $1 + 6 = 7, 2 + 3 = 5$. So 1 and 6 are the numbers we are looking for.

Step 3. We write $7x = 1x + 6x$ so

$$2x^2 + 7x + 3 = 2x^2 + x + 6x + 3.$$

Step 4.

$$2x^2 + x + 6x + 3 = (2x^2 + x) + (6x + 3) \quad \text{Factor by Grouping}$$

$$= x(2x + 1) + 3(2x + 1)$$

$$= (x + 3)(2x + 1).$$

We can check to see if we factored correctly by distributing our answer. We can use the FOIL method learned previously to check if the factored binomials give us the original trinomial $2x^2 + 7x + 3$.

First	$2x^2$
Outer	$+1x$
Inner	$+6x$
Last	$+3$

$$= 2x^2 + 1x + 6x + 3 = 2x^2 + 7x + 3$$

Example 14.1. Factor the given polynomial by the ac-Method.

a) $2x^2 + 15x - 27$:

Step 1. The product of $ac = (2)(-27) = -54$.

Stepv 2. Now we need to find two integers whose product is -54. We can list all the possibilities:

$$(-1)(54), \quad (-2)(27), \quad (-3)(18), \quad (-6)(9),$$

$$(1)(-54), \quad (2)(-27), \quad (3)(-18), \quad (6)(-9)$$

and calculate the sum of each pair. Only integers -3 and 18 add up to 15.

Step 3. We can rewrite the middle term $15x = -3x + 18x$. So $2x^2 + 15x - 27 = 2x^2 - 3x + 18x - 27$.

Step 4. We factor by grouping.

$$2x^2 + 15x - 27 = 2x^2 - 3x + 18x - 27$$

$$= (2x^2 - 3x) + (18x - 27)$$

$$= x(2x - 3) + 9(2x - 3)$$

$$= (x + 9)(2x - 3).$$

b) $12x^2 - 11x + 2$:

Step 1. The product of $ac = (12)(2) = 24$.

Step 2. We need to find two integers whose product is 24 and whose sum is -11. We list all pairs of factors of 24:

$$(1)(24), \quad (2)(12), \quad (3)(8), \quad (4)(6),$$

$$(-1)(-24), \quad (-2)(-12), \quad (-3)(-8), \quad (-4)(-6).$$

The pair -3 and -8 will have a sum -11.

Step 3. We rewrite the middle term $-11x = (-3x) + (-8x)$.

Step 4. Then we can finish the factoring.

$$12x^2 - 11x + 2 = 12x^2 - 3x + (-8x) + 2$$

$$= (12x^2 - 3x) + (-8x + 2)$$

$$= 3x(4x - 1) + (-2)(4x - 1)$$

$$= (3x - 2)(4x - 1).$$

Note that when we rewrite the middle term, we write it as a sum (even if the second term is negative). This is in order to be able to group without worrying about subtraction. Since otherwise the grouping step would look like this: $12x^2 - 3x - 8x + 2 = (12x^2 - 3x) - (8x - 2)$ (note the subtraction of 2). Also, note that we factored out (-2) in the second to the last step. This was in order to make sure that $(4x - 1)$ was a common factor.

c) $3x^2 + 4x - 2$:

The product of $ac = (3)(-2) = -6$, and this number factors as:

$$(-1)(6), \quad (-2)(3), \quad (1)(-6), \quad (2)(-3).$$

It is clear that none of pairs in the list will give a sum 4. This means that the polynomial $3x^2 + 4x - 2$ cannot be factored into two binomials (using integers). We call it a prime polynomial.

Factoring Trinomials $x^2 + bx + c$

In the special case when $a = 1$, the AC-method still works. For example, to factor $x^2 - 6x + 5$, we first compute $ac = (1)(5) = 5$. Then we need to find two numbers whose product is 5 and whose sum is -6. Since $(-1)(-5) = 5$ and $(-1) + (-5) = -6$, by the grouping method we have:

$$x^2 - 6x + 5 = x^2 - 1x + (-5x) + 5$$

$$= (x^2 - 1x) + (-5x + 5)$$

$$= x(x - 1) + (-5)(x - 1)$$

$$= (x - 5)(x - 1).$$

Now let's observe the result. The result has the form $(x + \boxed{-5})(x + \boxed{-1})$, and the two numbers in the two boxes are just the two numbers we get to rewrite the coefficient of the middle term -6, that is -1 and -5.

This example shows that to factor $x^2 + bx + c$, the grouping method can be simplified. We can directly write out the factored form of the polynomial once we know the two numbers that multiply to ac and add to b. In other words, $x^2 + bx + c$ is factored as $(x + \square)(x + \square)$, the product of the two numbers in the boxes being $ac = (1)(c) = c$ and the sum of the two numbers in the boxes being b.

Example 14.2. Factor the given trinomial.

a) $x^2 + 7x + 10$:

We need to find two numbers whose product is $ac = c = 10$, and whose sum is 7. Number 10 is a product of the following two numbers:

$$(1)(10), \quad (2)(5), \quad (-1)(-10), \quad (-2)(-5).$$

The pair 2 and 5 gives a sum 7, therefore the trinomial can be factored as:

$$x^2 + 7x + 10 = (x + 2)(x + 5).$$

b) $t^2 + 4t - 12$:

We need to find two numbers whose product is $ac = c = -12$ whose sum is 4. The number -12 is a product of the following two numbers:

$$(1)(-12), \quad (2)(-6), \quad (3)(-4)$$

$$(-1)(12), \quad (-2)(6), \quad (-3)(4).$$

The pair -2 and 6 gives a sum 4, therefore the trinomial can be factored as:

$$t^2 + 4t - 12 = (t + (-2))(t + 6) = (t - 2)(t + 6).$$

c) $x^2 - 3x - 24$:

We need to find two numbers whose product is $ac = c = -24$ whose sum is -3. The number -24 can be factored as:

$$(1)(-24), \quad (2)(-12), \quad (3)(-8), \quad (4)(-6)$$

$$(-1)(24), \quad (-2)(12), \quad (-3)(8), \quad (-4)(6).$$

Since none of the pairs in the list adds up to -3, the trinomial cannot be factored as a product of two binomials. This is a prime polynomial.

Mixed Factoring

So far, we have explained the basic techniques of factoring polynomials. Here is the guideline we can follow to select the right method to factor a given polynomial completely.

> **Guidelines to factoring a polynomial completely**
>
> Step 1. Factor out the GCF from all terms if possible.
>
> Step 2. Count the number of terms of the polynomial: if the polynomial has two terms, try the formula of difference of two squares; if the polynomial has three terms, try the AC-method; if the polynomial has four terms, try the grouping method.
>
> Step 3. Check to see if the factors themselves can be factored. If the answer is yes, then factor them completely using the methods in step 2.

Example 14.3. Factor the given polynomial completely.

a) $3x^2 - 12$:

$$3x^2 - 12 = 3(x^2 - 4) \qquad \text{Factor out the GCF 3}$$

$$= 3(x + 2)(x + (-2)) \quad \text{Factor the difference of two squares } x^2 - 4$$

$$= 3(x + 2)(x - 2)$$

b) $4x^3 - 20x^2 + 24x$:

$$4x^3 - 20x^2 + 24x = 4x(x^2 - 5x + 6) \qquad \text{Factor out the GCF } 4x$$

$$= 4x(x + (-2))(x + (-3)) \qquad \text{Factor the trinomial } x^2 - 5x + 6$$

$$= 4x(x - 2)(x - 3)$$

c) $-10z^2 - 4z + 6$:

$$-10z^2 - 4z + 6 = -2(5z^2 + 2z - 3) \qquad \text{Factor out the opposite of the GCF } -2$$

$$= -2(5z^2 - 3z + 5z - 3) \qquad \text{Factor the trinomial } 5z^2 + 2z - 3$$

$$= -2[(5z^2 - 3z) + (5z - 3)]$$

$$= -2[z(5z - 3) + 1(5z - 3)]$$

$$= -2(z + 1)(5z - 3)$$

d) $x^3 - 7x^2 - 4x + 28$:

$$x^3 - 7x^2 - 4x + 28 = (x^3 - 7x^2) + (-4x + 28) \qquad \text{Factor by grouping}$$

$$= x^2(x - 7) + (-4)(x - 7)$$

$$= (x^2 - 4)(x - 7)$$

$$= (x + 2)(x + (-2))(x - 7) \qquad \text{Factor } x^2 - 4$$

$$= (x + 2)(x - 2)(x - 7)$$

e) $30x^2 + 10x^4 - 280$

$$
\begin{aligned}
30x^2 + 10x^4 - 280 &= 10x^4 + 30x^2 - 280 &&\text{Reorder in decreasing powers of the variable}\\
&= 10(x^4 + 3x^2 - 28) &&\text{Factor out the GCF } 10\\
&= 10(y^2 + 3y - 28) &&\text{Write } y = x^2 \text{ to recognize as a quadratic expression}\\
&= 10(y + 7)(y - 4)\\
&= 10(x^2 + 7)(x^2 - 4) &&\text{Replace } y \text{ with } x^2 \text{ and look to see if anything can b}\\
&= 10(x^2 + 7)(x + 2)(x - 2)
\end{aligned}
$$

Exit Problem

Factor completely: $8x^2 - 10x + 3$

Chapter 15

Equations and Their Solutions

An **equation** is an expression that *is equal* to another expression.

Example 15.1. Examples of equations:

a) $x - 4 = 6$

b) $5x - 6 = 4x + 2$

c) $6x - 30 = 0$

d) $x^2 + 3x - 4 = 0$

e) $3x^2 - 2x = -1$

f) $x^3 + x^2 + x + 1 = 0$

g) $2x - 5 > 3$ is *not* an equation. It is an inequality and will be discussed in chapter 21.

> A **solution of an equation** is any value of the variable that
> satisfies the equality, that is, it makes the Left Hand Side
> (LHS) and the Right Hand Side (RHS) of the equation the
> same value.

To **solve an equation** is to **find the solution(s)** for that equation. The
method to solve an equation depends on the kind of equation at hand. We will
study how to:

- solve linear equations in chapters 16 and 17

- solve quadratic equations in chapter 20

Example 15.2. Solutions of equations:

a) A solution for $x - 4 = 6$ is $x = 10$ because the LHS evaluated at $x = 10$ is
   $10 - 4 = 6$ which is equal to the RHS.

b) A solution for $5x - 6 = 4x + 2$ is $x = 8$ because the LHS evaluated at
   $x = 8$ is $5(8) - 6 = 40 - 6 = 34$ and the RHS evaluated at $x = 8$ is
   $4x + 2 = 4(8) + 2 = 32 + 2 = 34$, and they are equal!

So, given a value of x, we can check if it is a solution or not by evaluating
simultaneously the LHS and RHS of an equation. If they are **equal**, then the
value is **a solution**. If they are **not equal**, then the value is **not a solution**.

Example 15.3. a) Is $x = 2$ a solution of the equation

$$-4x + 8 + x = 5 - 2x + 1 \qquad ?$$

The LHS evaluated at $x = 2$ is $-4(2) + 8 + 2 = -8 + 8 + 2 = 2$.
The RHS evaluated at $x = 2$ is $5 - 2(2) + 1 = 5 - 4 + 1 = 5 + (-4) + 1 = 1 + 1 = 2$.
Since they are equal, then we say that $x = 2$ is a solution for the given
equation.

b) Is $x = -1$ a solution of the equation

$$x^2 + 4x = -3x + 8 \qquad ?$$

The LHS evaluated at $x = -1$ is $(-1)^2 + 4(-1) = 1 + (-4) = -3$.
The RHS evaluated at $x = -1$ is $-3(-1) + 8 = 3 + 8 = 11$.
Since $-3 \neq 11$, then we say that $x = -1$ is **not a solution** for the given equation.

c) Is $x = -2$ a solution to

$$x^2 - 2x + 1 = 3x^2 + 2x + 1 \qquad ?$$

The LHS evaluated at $x = -2$ is $(-2)^2 - 2(-2) + 1 = 4 + 4 + 1 = 9$.
The RHS evaluated at $x = -2$ is $3(-2)^2 + 2(-2) + 1 = 3 \cdot 4 - 4 + 1 = 12 - 4 + 1 = 9$. Since $9 = 9$, the LHS=RHS and $x = -2$ is a solution to the equation.

Exit Problem

Check: Is $x = -6$ a solution of the equation

$$10 + 10x = 13 + 6x + 1?$$

Chapter 16

Solving Linear Equations

The kind of the equation will determine the method we use to solve it. We will first discuss **linear equations**. These are equations that only contain the first power of a variable and nothing higher.

Example 16.1. Examples of linear equations:

a) $x - 4 = 6$ is a linear equation.

b) $5x - 6 = 4x + 2$ is a linear equation.

c) $x^2 - 2x + 1 = 0$ is **not** a linear equation, since the variable x is to the second power. This is a quadratic equation which we will study in chapter 20.

   Critical Observation: We can add or subtract anything from an equation as long as we do it to both sides at the same time. This is a very essential tool to solve linear equations. It will help us **isolate** the variable on one side of the equation and the numbers on the other side of the equation.

$$\text{If} \quad a = b \quad \text{then} \quad a + c = b + c.$$
$$\text{If} \quad a = b \quad \text{then} \quad a - c = b - c.$$

Example 16.2. Isolate the variable in the given equation:

a) $x - 4 = 6$

Here we add 4 to both sides of the equation to get

$$x - 4 + 4 = 6 + 4$$

which has the effect of isolating the x on one side of the equation and the numbers on the other since upon simplifying we see that

$$x = 10.$$

It can be helpful to write this in a vertical form:

$$x - 4 = 6$$

$$+4 \quad +4$$

$$\Longrightarrow x = 10$$

b) $x + 7 = -2$. Here we add -7 from both sides because it will have the effect of isolating the x: (vertically written)

$$x + 7 = -2$$

$$+ -7 \quad -7$$

$$\Longrightarrow x = -9$$

c) $5x - 6 = 4x + 2$

Here x appears on both sides of the equation. If we subtract one of the terms from both sides, it will have the effect of isolating the x on one side.

We have a choice. We will subtract $4x$ from both sides so that an x is on the LHS. The alternative would have been to subtract $5x$ which would have left us with $-x$ on the RHS (this would be somewhat inconvenient). We have (vertically written)

$$5x - 6 = 4x + 2$$

$$-4x \quad\ - 4x$$

$$\implies x - 6 = 2$$

$$+6 \quad + 6$$

$$\implies x = 8$$

Note that each solution can be checked by plugging the number found into the original equation.

Example 16.3. Solve:

a) $17 - (4 - 2x) = 3(x + 4)$

To solve this equation, we first need to remove all parentheses and combine any like terms.

$$17 - (4 - 2x) = 3(x + 4)$$

$$\implies 17 - 4 + 2x = 3x + 12$$

$$\implies 13 + 2x = 3x + 12$$

Following the example above, the solution is found (by subtracting $2x$ from both sides and subtracting 12 from both sides) to be $x = 1$. Now, we can check if our work is correct by substituting $x = 1$ in the original equation and seeing whether the RHS and LHS yield the same value:

RHS: $17 - (4 - 2x) = 17 + (-1)(4 + (-2x)) = 17 + (-1)(4 + (-2 \cdot 1)) = 17 + (-1)(4 + (-2)) = 17 + (-1)(2) = 17 + (-2) = 15$

LHS: $3(x + 4) = 3(1 + 4) = 3(5) = 15$.

Since both values are equal, our solution of $x = 1$ is correct.

Critical Observation: We can multiply or divide an equation by any non-zero number as long as we do it to both sides at the same time. This is a very essential tool to solve linear equations where the coefficient of the variable is not 1.

$$
\begin{aligned}
&\text{If} \quad a = b \quad \text{then} \quad a \times c = b \times c. \\
&\text{If} \quad a = b \quad \text{then} \quad \frac{a}{c} = \frac{b}{c} \qquad \text{when } c \neq 0.
\end{aligned}
$$

Example 16.4. a) $6x = 42$

$$6x = 42$$

$$\implies \frac{6x}{6} = \frac{42}{6}$$

$$\implies x = 7$$

b) $-4x - 30 = 0$

In this example we will first isolate the 'x-term' which is $4x$ before isolating x.

$$-4x - 30 = 0$$

$$+30 \qquad + 30$$

$$\rule{3cm}{0.4pt}$$

$$\implies -4x = 30$$

$$\implies \frac{-4x}{-4} = \frac{30}{-4}$$

$$\implies x = -\frac{15}{2}$$

c) $2x = \dfrac{1}{4}$

$$2x = \frac{1}{4}$$

$$\implies \frac{2x}{2} = \frac{1}{4} \div 2$$

$$\implies x = \frac{1}{4} \cdot \frac{1}{2}$$

$$\implies x = \frac{1 \cdot 1}{4 \cdot 2}$$

$$\implies x = \frac{1}{8}$$

Note that dividing by 2 on both sides of the equation is the same as multiplying by $\dfrac{1}{2}$. So, we can rewrite the solution like this:

$$2x = \frac{1}{4}$$

$$\implies \frac{1}{2} \cdot 2x = \frac{1}{2} \cdot \frac{1}{4}$$

$$\implies \frac{1 \cdot 2x}{2} = \frac{1 \cdot 1}{4 \cdot 2}$$

$$\implies x = \frac{1}{8}$$

Generally, multiplying a number by its reciprocal results in 1:

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = 1.$$

Let's use this fact in the next example.

d) $\dfrac{2x}{3} = \dfrac{5}{6}$

$$\frac{2x}{3} = \frac{5}{6}$$

$$\implies \frac{2}{3} \cdot x = \frac{5}{6}$$

$$\implies \frac{2}{3} \cdot x = \frac{5}{6}$$

$$\implies \frac{3}{2} \cdot \frac{2}{3} \cdot x = \frac{3}{2} \cdot \frac{5}{6}$$

$$\implies x = \frac{3 \cdot 5}{2 \cdot 6}$$

$$\implies x = \frac{5}{2 \cdot 2}$$

$$\implies x = \frac{5}{4}$$

e) $\dfrac{x}{5} + 3 = 6$

$$\frac{x}{5} + 3 = 6$$

$$-3 \quad -3$$

$$\Longrightarrow \frac{x}{5} = 3$$

$$\Longrightarrow 5 \cdot \frac{x}{5} = 5 \cdot 3$$

$$\Longrightarrow x = 15$$

f) $5x - 6 = 2x + 3$

$$5x - 6 = 2x + 3$$

$$+6 \quad\quad + 6$$

$$\Longrightarrow 5x = 2x + 9$$

$$\Longrightarrow -2x \quad\quad - 2x$$

$$\Longrightarrow 3x = 9$$

$$\Longrightarrow \frac{3x}{3} = \frac{9}{3}$$

$$\Longrightarrow x = 3$$

g) $-3(x-1) = 4(x+2) + 2$

We first remove the parentheses and collect like terms.

$$-3(x-1) = 4(x+2) + 2$$

$$\implies -3x + 3 = 4x + 8 + 2$$

$$\implies -3x + 3 = 4x + 10$$

Now we proceed to solve the linear equation by isolating the variable:

$$-3x + 3 = 4x + 10$$

$$-3 \qquad -3$$

$$\implies -3x = 4x + 7$$

$$-4x \qquad -4x$$

$$\implies -7x = 7$$

$$\implies \frac{-7x}{-7} = \frac{7}{-7}$$

$$\implies x = -1$$

h) $10 - 3x = -2(x-1)$

$$10 - 3x = -2(x-1)$$

$$\implies 10 - 3x = -2x + 2$$

$$-10 \qquad \quad -10$$

$$\overline{}$$

$$\implies -3x = -2x - 8$$

$$+2x \quad + 2x$$

$$\overline{}$$

$$\implies -x = -8$$

$$\implies x = 8$$

Exit Problem

Solve: $5y - (7 - 2y) = 2(y + 4)$

Chapter 17

Solving Linear Equations, Decimals, Rationals

In this chapter we look at certain types of linear equations, those including decimal coefficients or rational coefficients. The reason why we discuss these separately is because we can "get rid" of the decimal numbers or denominators in the equation by performing a simple trick.

Recall from page 22 how we multiply decimal numbers by powers of 10 (that is $10, 100, 1000, \ldots$).

Example 17.1. Examples of multiplying by powers of 10:

a) $0.05 \times 100 = 5$

b) $2.23 \times 10 = 22.3$

c) $0.7 \times 100 = 70$

d) $0.2 \times 10 = 2$

Let's look at the following linear equation with decimal coefficients:

$$0.02y + 0.1y = 2.4$$

Step 1. Look at all the decimal numbers in the given equation: 0.02, 0.1 and 2.4.

Step 2. Pick the number(s) with the **most** decimal place(s) and count how many: 0.02 has **two** decimal places.

Step 3. Multiply both sides of the equation by 100 (1 and **two** zeros) because the most number of decimal places is **two**.

Multiply both sides by 100: $\qquad 100 \times (0.02y + 0.1y) = 100 \times (2.4)$

Distribute: $\qquad 100 \times 0.02y + 100 \times 0.1y = 100 \times 2.4$

$$\implies 2y + 10y = 240$$

Step 4. Proceed to solve the linear equation as usual.

$$2y + 10y = 240$$

$$\implies 12y = 240$$

$$\implies y = \frac{240}{12}$$

$$\implies y = \frac{\cancel{12} \cdot 20}{\cancel{12}}$$

$$\implies y = 20$$

Example 17.2. Solve the given equation:

a) $1.4 = 0.2x + 4$

Step 1. Look at all the decimal numbers in the given equation: 1.4 and 0.2.

Step 2. Pick the number(s) with the **most** decimal places and count how many: 1.4 and 0.2 both have **one** decimal place.

Step 3. Multiply both sides of the equation by 10 (1 and **one** zero) because the most number of decimal places is **one**.

Multiply both sides by 10: $\quad 10 \times (1.4) = 10 \times (0.2x + 4)$

Distribute: $\qquad\qquad\qquad 10 \times 1.4 = 10 \times 0.2x + 10 \times 4$

$$14 = 2x + 40$$

Step 4. Proceed to solve the linear equation as usual.

$$14 = 2x + 40$$

$$-40 \qquad\quad -40$$

$$\implies -26 = 2x$$

$$\implies \frac{-26}{2} = \frac{2x}{2}$$

$$\implies -13 = x$$

b) $0.7 + 0.28x = 1.26$

Step 1. Look at all the decimal numbers in the given equation: 0.7, 0.28 and 1.26.

Step 2. Pick the number(s) with the **most** decimal places and count how many: 0.28 and 1.26 both have **two** decimal places.

Step 3. Multiply both sides of the equation by 100 (1 and **two** zeros) because the most number of decimal places is **two**.

Multiply both sides by 100: $\quad 100 \times (0.7 + 0.28x) = 100 \times (1.26)$

Distribute: $\qquad\qquad\qquad 100 \times 0.7 + 100 \times 0.28x = 100 \times 1.26$

$$70 + 28x = 126$$

Step 4. Proceed to solve the linear equation as usual.

$$70 + 28x = 126$$

$$-70 \qquad -70$$

$$\implies 28x = 56$$

$$\implies \frac{28x}{28} = \frac{56}{28}$$

$$\implies x = 2$$

c) $0.5x - 0.235 = 0.06$

Step 1. Look at all the decimal numbers in the given equation: 0.5, 0.235 and 0.06.

Step 2. Pick the number(s) with the **most** decimal places and count how many: 0.235 has **three** decimal places.

Step 3. Multiply both sides of the equation by 1000 (1 and **three** zeros) because the most number of decimal places is **three**.

Multiply both sides by 1000: $\qquad 1000 \times (0.5x - 0.235) = 1000 \times (0.06)$

Distribute $\qquad 1000 \times 0.5x - 1000 \times 0.235 = 1000 \times 0.06$

$$500x - 235 = 60$$

Step 4. Proceed to solve the linear equation as usual

$$500x - 235 = 60$$

$$+235 \qquad +235$$

$$\rule{180pt}{0.5pt}$$

$$\implies 500x = 295$$

$$\implies \frac{500x}{500} = \frac{295}{500}$$

$$\implies x = \frac{\cancel{5} \cdot 59}{\cancel{5} \cdot 100}$$

$$\implies x = \frac{59}{100}$$

In the last example, we could have proceeded as follows. Writing the equation using fractions gives

$$\frac{5}{10}x - \frac{235}{1000} = \frac{6}{100}.$$

Note that in the above example we multiplied by 1000 which is the least common denominator (the denominators being $10, 1000$, and 100). We get

$$\frac{1000 \cdot 5}{10}x - \frac{1000 \cdot 235}{1000} = \frac{1000 \cdot 6}{100}.$$

Simplifying gives

$$100 \cdot 5x - 235 = 10 \cdot 6, \text{ or equivalently, } 500x - 235 = 60.$$

This brings us to Step 4 above.

We can use this method to solve a linear equation involving fractions. Lets look at a couple of examples.

Example 17.3. Solving linear equations with rational coefficients

a) Solve $\dfrac{1}{2} - \dfrac{3}{5}x = \dfrac{1}{6}$.

We could just treat this as in the last section but the arithmetic involves fractions. Subtract $\frac{1}{2}$ from both sides (noting that $\frac{1}{6} - \frac{1}{2} = \frac{1}{6} - \frac{3}{6} = -\frac{2}{6} = -\frac{1}{3}$)

$$-\frac{3}{5}x = -\frac{1}{3}.$$

Now multiplying both sides by $-\frac{5}{3}$ gives

$$x = \left(-\frac{5}{3}\right)\left(-\frac{1}{3}\right) = \frac{5}{9}.$$

But we could also proceed by clearing fractions:

Step 1. List the denominators. The denominators are $2, 5,$ and 6.

Step 2. Find the least common denominator. The least common denominator is 30.

Step 3. Multiply both sides of the equation by the LCD (30) and simplify:

$$\frac{30 \cdot 1}{2} - \frac{30 \cdot 3}{5}x = \frac{30 \cdot 1}{6} \implies 15 - 18x = 5.$$

Step 4. Solve as usual. We subtract 15 from both sides:

$$-18x = -10.$$

Dividing by -18 gives

$$x = \frac{-10}{-18} = \frac{5}{9}.$$

b) Solve $2 - \dfrac{5x}{3} = \dfrac{3x}{5} - \dfrac{1}{6}$.

Step 1. Identify the denominators $3, 5,$ and 6.

Step 2. Find the least common denominator 30.

Step 3. Multiply both sides of the equation by 30 and simplify.

$$30 \cdot 2 - \frac{30 \cdot 5x}{3} = \frac{30 \cdot 3x}{5} - \frac{30 \cdot 1}{6} \implies 60 - 50x = 18x - 5.$$

Step 4. Solve as usual. Adding $50x$ to both sides gives

$$60 = 68x - 5.$$

Adding 5 to both sides gives

$$65 = 68x.$$

Dividing by 68 gives

$$x = \frac{65}{68}.$$

Exit Problem

Solve: $0.03x + 2.5 = 4.27$

Chapter 18

Word Problems for Linear Equations

Word problems are important applications of linear equations. We start with examples of translating an English sentence or phrase into an algebraic expression.

Example 18.1. Translate the phrase into an algebraic expression:

a) Twice a variable is added to 4.

   Solution: We call the variable x. Twice the variable is $2x$. Adding $2x$ to 4 gives:
   $$4 + 2x$$

b) Three times a number is subtracted from 7.

   Solution: Three times a number is $3x$. We need to subtract $3x$ from 7. This means:
   $$7 - 3x$$

c) 8 less than a number.

Solution: The number is denoted by x. 8 less than x mean, that we need to subtract 8 from it. We get:

$$x - 8$$

For example, 8 less than 10 is $10 - 8 = 2$.

d) Subtract $5p^2 - 7p + 2$ from $3p^2 + 4p$ and simplify.

Solution: We need to calculate $3p^2 + 4p$ minus $5p^2 - 7p + 2$:

$$(3p^2 + 4p) - (5p^2 - 7p + 2)$$

Simplifying this expression gives:

$$(3p^2+4p)-(5p^2-7p+2) \quad = \quad 3p^2+4p-5p^2+7p-2 \quad = \quad -2p^2+11p-2$$

e) The amount of money given by x dimes and y quarters.

Solution: Each dime is worth 10 cents, so that this gives a total of $10x$ cents. Each quarter is worth 25 cents, so that this gives a total of $25y$ cents. Adding the two amounts gives a total of

$$10x + 25y \text{ cents or } .10x + .25y \text{ dollars}$$

Now we deal with word problems that directly describe an equation involving one variable, which we can then solve.

Example 18.2. Solve the following word problems:

a) Five times an unknown number is equal to 60. Find the number.

Solution: We translate the problem to algebra:

$$5x = 60$$

We solve this for x:

$$x = \frac{60}{5} = 12$$

b) If 5 is subtracted from twice an unknown number, the difference is 13. Find the number.

Solution: Translating the problem into an algebraic equation gives:

$$2x - 5 = 13$$

We solve this for x. First, add 5 to both sides.

$$2x = 13 + 5, \text{ so that } 2x = 18$$

Dividing by 2 gives $x = \frac{18}{2} = 9$.

c) A number subtracted from 9 is equal to 2 times the number. Find the number.

Solution: We translate the problem to algebra.

$$9 - x = 2x$$

We solve this as follows. First, add x:

$$9 = 2x + x \text{ so that } 9 = 3x$$

Then the answer is $x = \frac{9}{3} = 3$.

d) Multiply an unknown number by five is equal to adding twelve to the unknown number. Find the number.

Solution: We have the equation:

$$5x = x + 12.$$

Subtracting x gives

$$4x = 12.$$

Dividing both sides by 4 gives the answer: $x = 3$.

e) Adding nine to a number gives the same result as subtracting seven from three times the number. Find the number.

Solution: Adding 9 to a number is written as $x + 9$, while subtracting 7 from three times the number is written as $3x - 7$. We therefore get the equation:

$$x + 9 = 3x - 7.$$

We solve for x by adding 7 on both sides of the equation:

$$x + 16 = 3x.$$

Then we subtract x:

$$16 = 2x.$$

After dividing by 2, we obtain the answer $x = 8$.

The following word problems consider real world applications. They require to model a given situation in the form of an equation.

Example 18.3. Solve the following word problems:

a) Due to inflation, the price of a loaf of bread has increased by 5%. How much does the loaf of bread cost now, when its price was $2.40 last year?

Solution: We calculate the price increase as $5\% \cdot \$2.40$. We have

$$5\% \cdot 2.40 = 0.05 \cdot 2.40 = 0.1200 = 0.12$$

We must add the price increase to the old price.

$$2.40 + 0.12 = 2.52$$

The new price is therefore $2.52.

b) To complete a job, three workers get paid at a rate of $12 per hour. If the total pay for the job was $180, then how many hours did the three workers spend on the job?

Solution: We denote the number of hours by x. Then the total price is calculated as the price per hour ($12) times the number of workers (3) times the number of hours (x). We obtain the equation

$$12 \cdot 3 \cdot x = 180.$$

Simplifying this yields
$$36x = 180.$$

Dividing by 36 gives
$$x = \frac{180}{36} = 5.$$

Therefore, the three workers needed 5 hours for the job.

c) A farmer cuts a 300 foot fence into two pieces of different sizes. The longer piece should be four times as long as the shorter piece. How long are the two pieces?

Solution: We denote by x the length of the shorter piece of fence. Then the longer piece has four times this length, that is, it is of length $4x$. The two pieces together have a total length of 300 ft. This gives the equation

$$x + 4x = 300.$$

Combining the like terms on the left, we get

$$5x = 300.$$

Dividing by 5, we obtain that

$$x = \frac{300}{5} = 60.$$

Therefore, the shorter piece has a length of 60 feet, while the longer piece has four times this length, that is 4×60 feet $= 240$ feet.

d) If 4 blocks weigh 28 ounces, how many blocks weigh 70 ounces?

Solution: We denote the weight of a block by x. If 4 blocks weigh 28, then a block weighs $x = \dfrac{28}{4} = 7$.

How many blocks weigh 70? Well, we only need to find $\dfrac{70}{7} = 10$. So, the answer is 10.

Note You can solve this problem by setting up and solving the fractional equation $\dfrac{28}{4} = \dfrac{70}{x}$. Solving such equations is addressed in chapter 24.

e) If a rectangle has a length that is three more than twice the width and the perimeter is 20 in, what are the dimensions of the rectangle?

Solution: We denote the width by x. Then the length is $2x + 3$. The perimeter is 20in on one hand and $2(\text{length})+2(\text{width})$ on the other. So we have

$$20 = 2x + 2(2x + 3).$$

Distributing and collecting like terms give

$$20 = 6x + 6.$$

Subtracting 6 from both sides of the equation and then dividing both sides of the resulting equation by 6 gives:

$$20 - 6 = 6x \implies 14 = 6x \implies x = \frac{14}{6}\text{in} = \frac{7}{3}\text{in} = 2\frac{1}{3}\text{in}.$$

f) If a circle has circumference 4in, what is its radius?

Solution: We know that $C = 2\pi r$ where C is the circumference and r is the radius. So in this case

$$4 = 2\pi r.$$

Dividing both sides by 2π gives

$$r = \frac{4}{2\pi} = \frac{2}{\pi}\text{in} \approx 0.63\text{in}.$$

g) The perimeter of an equilateral triangle is 60 meters. How long is each side?

Solution: Let x equal the side of the triangle. Then the perimeter is, on the one hand, 60, and on other hand $3x$. So $3x = 60$ and dividing both sides of the equation by 3 gives $x = 20$ meters.

h) If a gardener has $600 to spend on a fence which costs $10 per linear foot and the area to be fenced in is rectangular and should be twice as long as it is wide, what are the dimensions of the largest fenced in area?

Solution: The perimeter of a rectangle is $P = 2L + 2W$. Let x be the width of the rectangle. Then the length is $2x$. The perimeter is $P = 2(2x) + 2x = 6x$. The largest perimeter is $600/(\$10/\text{ft}) = 60$ ft. So $60 = 6x$ and dividing both sides by 6 gives $x = 60/6 = 10$. So the dimensions are 10 feet by 20 feet.

i) A trapezoid has an area of 20.2 square inches with one base measuring 3.2 in and the height of 4 in. Find the length of the other base.

Solution: Let b be the length of the unknown base. The area of the trapezoid is on the one hand 20.2 square inches. On the other hand it is $\frac{1}{2}(3.2+b)\cdot 4 = 6.4 + 2b$. So
$$20.2 = 6.4 + 2b.$$

Multiplying both sides by 10 gives
$$202 = 64 + 20b.$$

Subtracting 64 from both sides gives
$$138 = 20b$$

and dividing by 20 gives
$$b = \frac{138}{20} = \frac{69}{10} = 6.9 \text{ in.}$$

Exit Problem

Write an equation and solve: A car uses 12 gallons of gas to travel 100 miles. How many gallons would be needed to travel 450 miles?

Chapter 19

Rewriting Formulas

Michael Faraday (1791 - 1867) was an English scientist who contributed to the fields of electromagnetism and electrochemistry. His mathematical abilities, however, were limited and so he mainly relied on expressing his ideas in clear and simple writing. Later, the scientist James Clerk Maxwell came along and took the work of Faraday (and others), and summarized it in a set of equations known as "Maxwell's Equations". His equations are accepted as the basis of all modern electromagnetic theory and take on many different mathematical forms.

The language of mathematics is powerful. It is a language which has the ability to express relationships and principles *precisely* and *succinctly*. Faraday was a brilliant scientist who made history-making discoveries yet they were not truly appreciated until Maxwell was able to translate them into a workable language, that of mathematics.

Definition of a Formula

A formula is a mathematical relationship expressed in symbols.

For example let's consider Einstein's $E = mc^2$ (arguably the most famous formula in the world). This formula is an equation which describes the relationship between the energy a body transmits in the form of radiation (the E) and its mass (the m) along with the speed of light in vacuum (the c). It says that a body of mass m emits energy of the amount E which is precisely equal to mc^2. It can also say that if a body emits an amount of energy E, then its mass m must be E/c^2. We can say this because we solved $E = mc^2$ for m, that is, we

rewrote the formula in terms of a specific variable.

Example 19.1. More familiar formulas:

a) $A = lw$ Area of a rectangle $=$ length $\cdot$ width.

b) $C = 2\pi r$ Circumference of a circle $= 2 \cdot \pi \cdot$ radius.

c) $S = d/t$ Speed $=$ distance traveled $\div$ time.

We note that the formula $C = 2\pi r$ is "solved" for C since C is by itself on one side of the equation. Suppose we know the circumference of a circle C is equal to 5 inches and we wish to find its radius r. One way to do this is to begin by solving the formula $C = 2\pi r$ and then substituting the known values.

Begin with:

$$C = 2\pi r$$

and divide both sides by 2π to get

$$\frac{C}{2\pi} = r.$$

Next we substitute the value $C = 5$ and π to get

$$r = \frac{5}{2\pi} \approx 0.8 \text{ in rounded to the nearest tenth.}$$

So $r = 0.8$ inches when we round to the nearest tenth.

> **Solving an Equation for a Specified Variable**
>
> 1. Use the distributive property of multiplication (if possible).
>
> 2. Combine any like terms (if possible).
>
> 3. Get rid of denominators (if possible).
>
> 4. Collect all terms with the variable you wish to solve for one side of the equation. Do this by using the addition property of equality.
>
> 5. Use the distributive property and the multiplication property of equality to isolate the desired variable.

Example 19.2. Solve the following equations for the specific variable indicated in parenthesis.

a) $E = IR$ (for I)

Divide both sides by R to get $I = \dfrac{E}{R}$.

b) $PV = nRT$ (for R)

Divide both sides by nT to get $R = \dfrac{PV}{nT}$.

c) $S = \dfrac{T}{P}$ (for P)

Multiply each side by P to get $PS = T$ then divide each side by S to get $P = \dfrac{T}{S}$.

d) $V = \dfrac{MN}{4Z}$ (for N)

Multiply each side by $4Z$ to get $4ZV = MN$, then divide each side by M to get $N = \dfrac{4ZV}{M}$.

e) $d = a + b + c$ (for b)

Subtract a and c from both sides to get $b = d - a - c$.

f) $B = \dfrac{zI^2 p}{W}$ (for p)

Multiply each side by W to get $WB = zI^2 p$, then divide both sides by zI^2 to get $p = \dfrac{WB}{zI^2}$.

g) $P = S - c$ (for c)

Add c to both sides to get $c + P = S$, then subtract P from both sides to get $c = S - P$.

h) $A = EC + G$ (for C)

Subtract G from both sides to get $A - G = EC$, then divide both sides by E to get $C = \dfrac{A - G}{E}$.

i) $w = 2x - 3y$ (for x)

Add $3y$ to both sides to get $w + 3y = 2x$, then divide both sides by 2 to get $\dfrac{w + 3y}{2} = x$, and so, $x = \dfrac{w + 3y}{2}$.

Exit Problem

The formula for the area of a triangle is $A = \dfrac{1}{2}bh$. Solve this equation for b.

Chapter 20

Solving Quadratic Equations by Factoring

As mentioned earlier, solving equations is dependent on the type of equation at hand. You can review solving linear equations in chapter 16. This chapter will deal with **solving quadratic equations**. These are equations that contain the second power of a variable and nothing higher.

Example 20.1. Examples of quadratic equations:

a) $x^2 - 3x - 18 = 0$ is a quadratic equation.

b) $x^2 = 81$ is a quadratic equation.

c) $4x^2 - 36 = 0$ is a quadratic equation.

d) $x^3 - 27 = 0$ is **not** a quadratic equation, since it includes a third degree variable – we will not discuss solving such equations in this class.

Solving Quadratic Equations

Solving quadratic equations involves three basic steps. We will look at each of these steps as we proceed to solve $x^2 = 100$.

Step 1. **Standard Form**

We say that an equation is in **standard form** if all the terms are collected on one side of the equal sign, and there is only a 0 on the other side. For instance, equations a) and c) in the previous example are quadratic equations in standard form.

If we are working with an equation that is not in standard form, we can easily get it to the desired form by adding or subtracting terms from both sides.

For instance, the equation $x^2 = 100$ is not in standard form. To get it to standard form, we subtract 100 from both sides of the equation:

$$x^2 = 100$$

$$-100 \quad -100$$

$$\implies x^2 - 100 = 0$$

Note: If the equation at hand is already in standard form, then we skip this step.

Step 2. **Factor**

Now that all the terms are on one side of the equation, we factor the quadratic expression at hand. It is a good idea to review the factoring techniques introduced in chapters 12, 13 and 14.

$$x^2 - 100 = 0$$

$$\implies x^2 - (10)^2 = 0$$

$$\implies (x - 10)(x + 10) = 0$$

Step 3. **Use the Zero-Product Property**

> **Zero-Product Property**
>
> If $A \cdot B = 0$ then $A = 0$ or $B = 0$.

The reason why this property is helpful in solving quadratic equations is because after you have put your equation in standard form and factored, you are exactly in a situation where you can apply the Zero-Product Property.

$$(x - 10)(x + 10) = 0$$

translates into:

$$(x - 10) = 0 \text{ or } (x + 10) = 0$$

and now, both of the equations at hand are linear, and solving them is a matter of isolating the variable with some algebraic manipulations. Notice that now you have two linear equations, each will give its own solution. So, your original quadratic equation will have two solutions.

$$x - 10 = 0 \text{ gives } x = 10$$

and

$$x + 10 = 0 \text{ gives } x = -10$$

So, the solutions of the equation $x^2 = 100$ are $x = 10$ and $x = -10$.

Step 4. **Check (optional)**

It is not mandatory (unless the question specifically asks for it), but it is a good habit to check whether the solutions obtained are correct. To check if our solutions are correct, we need to substitute 10 and -10 into the **original equation** $x^2 = 100$.

We easily see that $(10)^2 = 100$ and $(-10)^2 = 100$, so both solutions work.

Example 20.2. Solve:

a) $5(x-2)(x+3) = 0$

Notice that this equation is already in standard form and factored. To apply the Zero-Product Property for

$$5(x-2)(x+3) = 0.$$

We first divide both sides of the equation by 5 (or multiply by $\dfrac{1}{5}$):

$$\frac{5(x-2)(x+3)}{5} = \frac{0}{5}.$$

This gives

$$(x-2)(x+3) = 0.$$

Now we apply the Zero-Product Property

$$(x-2) = 0 \text{ or } (x+3) = 0.$$

$$x - 2 = 0 \text{ gives } x = 2$$

and

$$x + 3 = 0 \text{ gives } x = -3.$$

So, the solutions of the equation $5(x-2)(x+3) = 0$ are $x = 2$ and $x = -3$.

We can easily check that $5(x-2)(x+3)$ evaluated at $x = 2$ gives 0 and also at $x = -3$ gives 0.

b) $(5x-4)(x-6) = 0$

Notice that this equation is already in standard form and factored, and we can directly apply the Zero-Product Property.

$$(5x-4) = 0 \text{ or } (x-6) = 0$$

$$5x - 4 = 0 \text{ gives } 5x = 4 \text{ and thus } x = \frac{4}{5}$$

and

$$x - 6 = 0 \text{ gives } x = 6.$$

So, the solutions of the equation $(5x - 4)(x - 6) = 0$ are $x = \dfrac{4}{5}$ and $x = 6$.

Example 20.3. Solve the given quadratic equations:

a) $x^2 - 3x - 18 = 0$

This is a quadratic equation in standard form.

Standard Form $\qquad\qquad\qquad\qquad x^2 - 3x - 18 = 0$

Factor $\qquad\qquad\qquad\qquad\qquad \implies (x - 6)(x + 3) = 0$

Zero-Product Property $\qquad \implies (x - 6) = 0 \text{ or } (x + 3) = 0.$

$$x - 6 = 0 \text{ gives } x = 6$$

and

$$x + 3 = 0 \text{ gives } x = -3.$$

So $x = 6$ or $x = -3$.

b) $x^2 = 81$

This is a quadratic equation **not** in standard form.

Standard Form $\qquad\qquad\qquad\qquad x^2 - 81 = 0$

Factor $\qquad\qquad\qquad\qquad\qquad \implies x^2 - (9)^2 = 0$

$$\implies (x-9)(x+9) = 0$$

Zero-Product Property $\qquad \implies (x-9) = 0 \text{ or } (x+9) = 0.$

$$x - 9 = 0 \text{ gives } x = 9$$

and

$$x + 9 = 0 \text{ gives } x = -9.$$

The solutions are $x = 9$ and $x = -9$.

c) $3x^2 - 27 = 0$

This is a quadratic equation in standard form.

Standard Form $\qquad\qquad\qquad\qquad 3x^2 - 27 = 0$

Factor $\qquad\qquad\qquad\qquad \implies 3 \cdot x^2 - 3 \cdot 9 = 0$

$$\implies 3(x^2 - 9) = 0$$

$$\implies 3(x^2 - 3^2) = 0$$

$$\implies 3(x-3)(x+3) = 0$$

Zero-Product Property $\qquad \implies (x-3) = 0 \text{ or } (x+3) = 0.$

$$x - 3 = 0 \text{ gives } x = 3$$

and

$$x + 3 = 0 \text{ gives } x = -3.$$

The solutions are $x = 3$ and $x = -3$.

d) $x^2 + 5x + 4 = 0$

Standard Form	$x^2 + 5x + 4 = 0$
Factor	$\implies (x + 4)(x + 1) = 0$
Zero-Product Property	$\implies (x + 4) = 0 \text{ or } (x + 1) = 0$

$$x + 4 = 0 \text{ gives } x = -4$$

and

$$x + 1 = 0 \text{ gives } x = -1.$$

So if $x^2 + 5x + 4 = 0$ then $x = -4$ or $x = -1$.

e) $x^2 = 7x$

Standard Form	$x^2 - 7x = 0$
Factor	$\implies x \cdot x - 7 \cdot x = 0$
	$\implies x(x - 7) = 0$
Zero-Product Property	$\implies x = 0 \text{ or } (x - 7) = 0$

$$x = 0 \text{ gives } x = 0$$

and

$$x - 7 = 0 \text{ gives } x = 7.$$

The solutions are $x = 0$ and $x = 7$.

f) $4x^2 + 20x = 0$

Standard Form $\qquad\qquad 4x^2 + 20x = 0$

Factor $\qquad\qquad \implies 4x \cdot x + 4x \cdot 5 = 0$

$\qquad\qquad \implies 4x(x + 5) = 0$

Zero-Product Property $\qquad \implies 4x = 0 \text{ or } (x + 5) = 0.$

$$4x = 0 \text{ gives } \frac{4x}{4} = \frac{0}{4} \text{ thus } x = 0$$

and

$$x + 5 = 0 \text{ gives } x = -5.$$

The solutions are $x = 0$ and $x = -5$.

g) $4x^2 - 25 = 0$

Standard Form	$4x^2 - 25 = 0$
Factor	$\implies (2x)^2 - (5)^2 = 0$
	$\implies (2x - 5)(2x + 5) = 0$
Zero-Product Property	$\implies (2x - 5) = 0$ or $(2x + 5) = 0.$

$$2x - 5 = 0 \text{ gives } 2x = 5 \text{ and thus } x = \frac{5}{2}$$

and

$$2x + 5 = 0 \text{ gives } 2x = -5 \text{ and thus } x = \frac{-5}{2}.$$

The solutions are $x = \dfrac{5}{2}$ or $x = -\dfrac{5}{2}$.

h) $3x^2 + 7x + 2 = 0$

Standard Form	$3x^2 + 7x + 2 = 0$
Factor	$\implies 3x^2 + 6x + x + 2 = 0$
	$\implies 3x \cdot x + 3x \cdot 2 + x + 2 = 0$
	$\implies 3x(x + 2) + (x + 2) = 0$
	$\implies (3x + 1)(x + 2) = 0$
Zero-Product Property	$\implies (3x + 1) = 0$ or $(x + 2) = 0.$

$$3x + 1 = 0 \text{ gives } 3x = -1 \text{ and thus } x = \frac{-1}{3}$$

and

$$x + 2 = 0 \text{ gives } x = -2.$$

The solutions are $x = -\dfrac{1}{3}$ and $x = -2$.

i) $x^2 = -2x - 1$

Standard Form	$\implies x^2 + 2x + 1 = 0$
Factor	$\implies (x + 1)(x + 1) = 0$
Zero-Product Property	$\implies (x + 1) = 0 \text{ or } (x + 1) = 0.$

$$x + 1 = 0 \text{ gives } x = -1$$

and

$$x + 1 = 0 \text{ gives } x = -1.$$

Since both solutions are the same, we say that we have a double solution $x = -1$.

j) $2x^2 - 32 = 0$

Standard Form	$\implies 2x^2 - 32 = 0$
Factor	$\implies 2 \cdot x^2 - 2 \cdot 16 = 0$
	$\implies 2(x^2 - 16) = 0$
	$\implies 2(x^2 - 4^2) = 0$
	$\implies 2(x - 4)(x + 4) = 0$
Divide by 2 on both sides	$\implies (x - 4)(x + 4) = 0$
Zero-Product Property	$\implies (x - 4) = 0$ or $(x + 4) = 0.$

$$x - 4 = 0 \text{ gives } x = 4$$

and

$$x + 4 = 0 \text{ gives } x = -4$$

The solutions are $x = 4$ and $x = -4$.

k) $3x^2 - 3x - 6 = 0$

Standard Form	$3x^2 - 3x - 6 = 0$
Factor	$\implies 3 \cdot x^2 - 3 \cdot x - 3 \cdot 2 = 0$
	$\implies 3(x^2 - x - 2) = 0$
	$\implies 3(x - 2)(x + 1) = 0$

Divide by 3 on both sides $\implies (x-2)(x+1) = 0$

Zero-Product Property $\implies (x-2) = 0$ or $(x+1) = 0.$

$$x - 2 = 0 \text{ gives } x = 2$$

and

$$x + 1 = 0 \text{ gives } x = -1.$$

The solutions are $x = 2$ and $x = -1$.

If a rectangle has area 15 square feet and its length is two feet less than its width then what are the dimensions of the rectangle.

Let x represent the width. Then the length is $x - 2$. The area, on the one hand is 15 square feet, and on the other hand is $l \times w$. So

$$15 = x(x-2).$$

We first distribute to get
$$15 = x^2 - 2x.$$

Now we put it in standard form by subtracting 15 from both sides:

$$x^2 - 2x - 15 = 0.$$

Now we factor so that we can use the Zero-Product Property:

$$(x-5)(x+3) = 0.$$

Then $x - 5 = 0$ or $x + 3 = 0$, which gives that $x = 5$ or $x = -3$. But x is the width of a rectangle and therefore it can not be negative. The only solution to this problem is $x = 5$ (and $x - 2 = 3$). So the dimensions of the rectangle are 5 feet by 3 feet.

Solving Quadratic Equations Of The Form $x^2 = A$

In the special case when a quadratic equation is of the form $x^2 = A$, we can resort to a short-cut to solve it. Be aware that this short-cut **only works in this special case**.

We first note that if $x^2 = 9$ then, by inspection, the solutions are 3 and -3. We see that in general to solve $x^2 = A$, we see that the solutions are $x = \sqrt{A}$ and $x = -\sqrt{A}$.

Notice that if we use the factoring method, the equation $x^2 = A$ becomes $x^2 - A = 0$ in standard form, and the factored form of the equation is $(x - \sqrt{A})(x + \sqrt{A}) = 0$ which gives the solutions $x = \sqrt{A}$ and $x = -\sqrt{A}$.

Example 20.4. Use the shortcut above to solve the given special quadratic equation:

a) $x^2 = 100$

Step 1. $\sqrt{100} = 10$

Step 2. The solutions are $x = 10$ and $x = -10$

b) $x^2 = 72$

Step 1. $\sqrt{72} = 6\sqrt{2}$

Step 2. The solutions are $x = 6\sqrt{2}$ and $x = -6\sqrt{2}$

c) $x^2 - 75 = 0$

Step 1. $\sqrt{75} = 5\sqrt{3}$

Step 2. The solutions are $x = 5\sqrt{3}$ and $x = -5\sqrt{3}$

d) The area of a square is 20 square feet. How long is each side?

Let x be the length of each side. Then $x^2 = 20$. So $x = \sqrt{20} = 2\sqrt{5}$. Note that it can not be $-2\sqrt{5}$ since x is a length. Each side is $2\sqrt{5}$ ft.

Application to the Pythagorean Theorem

The Pythagorean Theorem relates the lengths of the legs of a right triangle and the hypotenuse.

The Pythagorean Theorem: If a and b are the lengths of the legs of the right triangle and c is the length of the hypotenuse (the side opposite the right angle) as seen in this figure,
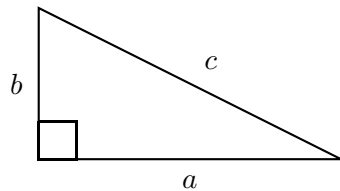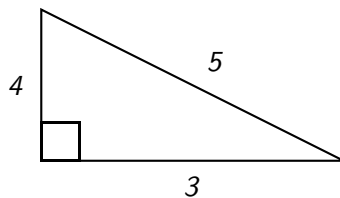

then

$$a^2 + b^2 = c^2.$$

Remark 20.5. In the theorem a and b are interchangeable.

Example 20.6. Consider the following right triangle.

It satisfies the conclusion of the Pythagorean Theorem since $3^2 + 4^2 = 5^2$ (check it).

Remark 20.7. Do not take the proportions of the triangles drawn seriously– what appears to be the shortest leg may not in fact be. These drawings are just cartoons that exhibit a relationship between the legs as they relate to the angles.

If we know any two sides of a right triangle we can find the third using the Pythagorean Theorem. We will look at a few examples.

Example 20.8. Find x if
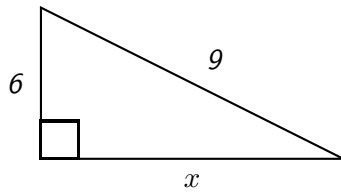
By the Pythagorean Theorem (noting that the length of the hypotenuse is 9 and so 81 belongs on one side of the equality)

$$x^2 + 6^2 = 9^2, \text{ or equivalently, } x^2 + 36 = 81$$

so that

$$x^2 = 81 - 36 \text{ or, equivalently, } x^2 = 45$$

Now since x represents a length, it is positive so that

$$x = \sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}.$$

Example 20.9. Find x given the following picture
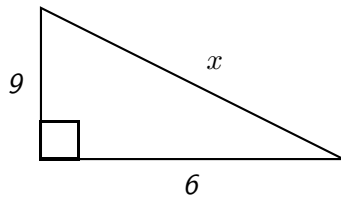
By the Pythagorean Theorem (noting that the length of the hypotenuse is x and so x^2 belongs on one side of the equality)

$$6^2 + 9^2 = x^2, \text{ or equivalently, } 36 + 81 = x^2$$

so that

$$x^2 = 117$$

Now since x represents a length, it is positive so that

$$x = \sqrt{117} = \sqrt{9 \cdot 13} = 3\sqrt{13}.$$

Example 20.10. Suppose you must reach a window of a house with a ladder so that the ladder meets the house 7 ft off the ground. Suppose also that the ladder must be 5 ft from the base of the house. How long must the ladder be?

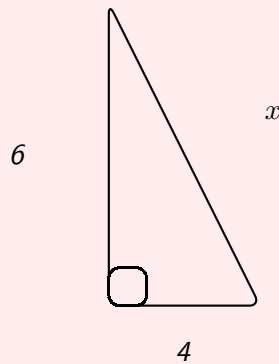This information leads to the triangle (not drawn to scale):
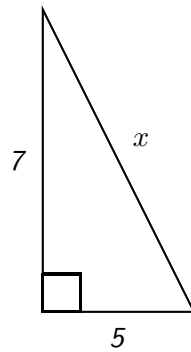
So by the Pythagorean Theorem says

$$5^2 + 7^2 = x^2$$

so that $x^2 = 74$. Therefore the ladder must be $\sqrt{74}$ ft long which is approximately 8.6 ft long.

Exit Problem

1. Solve: $x^2 - 5x = 6$

2. Solve: $16x^2 = 81$

3. Solve: $-24x = 10x^2$

4. Solve for x in the given right triangle.

Chapter 21

Linear Inequalities

In this section we solve linear inequalities.

Consider the inequality $2x - 1 > 2$. To solve this means to find all values of x that satisfy the inequality (so that when you plug in those values for x you get a true statement). For example, since $2 \cdot 5 - 1 = 9 > 2$, $x = 5$ is a solution and since $2 \cdot 1 - 1 = 1 \not> 2$, $x = 1$ is not a solution.
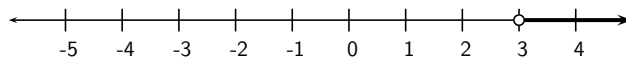
We can solve an inequality in much the same way as we solve an equality with one important exception:

> Multiplication or division by a negative number reverses the direction of the inequality.

For example $-3x > 9 \iff \dfrac{-3x}{-3} < \dfrac{9}{-3} \iff x < -3$. We see that $x = -10$ satisfies all of these inequalities and $x = 3$ satisfies none of them. We can graph the solution on the number line:
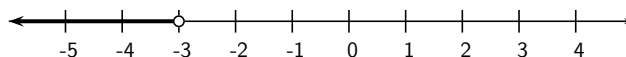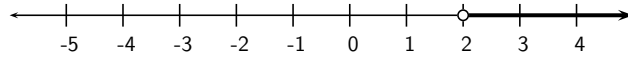

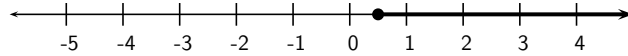But $3x > 9 \iff \dfrac{3x}{3} > \dfrac{9}{3} \iff x > 3$.

For instance, check that $x = 5$ (which is greater than 3) satisfies the inequality.

Example 21.1. Solve the given inequality and represent the solution on the number line:
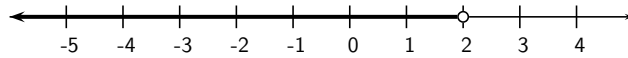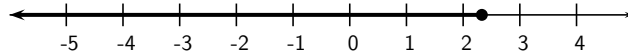
a) $5x > 10 \iff x > \dfrac{10}{5} \iff x > 2$

b) $-10x \leq -5 \iff x \geq \dfrac{-5}{-10} \iff x \geq \dfrac{1}{2}$

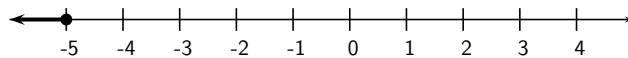c) $-x > -2 \iff x < \dfrac{-2}{-1} \iff x < 2$

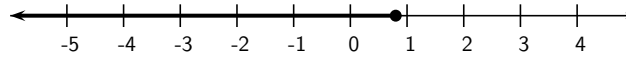d) $2 - x \geq 2x - 5 \iff 7 - x \geq 2x \iff 7 \geq 3x \iff \dfrac{7}{3} \geq x \,(\text{or } x \leq \dfrac{7}{3})$

e) $2 - 3x \geq -2x + 7 \iff -5 - 3x \geq -2x \iff -5 \geq x \,(\text{or } x \leq -5)$

f) $3(x-2)+5 \leq 5-2(x+1) \iff 3x-6+5 \leq 5-2x-2 \iff 3x-1 \leq$
$3-2x \iff 3x+2x \leq 3+1 \iff 5x \leq 4 \iff x \leq \frac{4}{5}$

Note: There is more than one way to do a problem. For example:

$-3x-2 < 1 \iff -3x < 2+1 \iff -3x < 3 \iff \dfrac{-3x}{-3} > \dfrac{3}{-3} \iff$
$x > -1$

or,

$-3x-2 < 1 \iff -2 < 1+3x \iff -2-1 < 3x \iff -3 < 3x \iff$
$-1 < x$ (which is the same as $x > -1$)
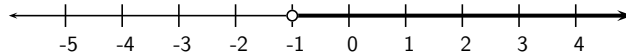
Exit Problem

Solve the inequality and show the graph of the solution:
$$7x+4 \leq 2x-6$$

Chapter 22

Simplifying, Multiplying and Dividing Rational Expressions

We recall that a rational number is one which can be written as a ratio $\dfrac{p}{q}$ where p and q are integers and $q \neq 0$. A **rational expression** (also called an **algebraic fraction**) is one which can be written as a ratio $\dfrac{P}{Q}$ where P and Q are polynomials and $Q \neq 0$. Just as we can write a number in simplest (or reduced) form, we can do the same for rational expressions.

The Fundamental Principle of Rational Expressions
For polynomials P, Q and R with $Q \neq 0$ and $R \neq 0$

$$\frac{P}{Q} = \frac{P \cdot R}{Q \cdot R}$$

> **To Simplify a Rational Expression**
>
> Step 1. Find the GCF of both numerator and denominator, if you can.
>
> Step 2. Factor completely both numerator and denominator.
>
> Step 3. Use the Fundamental Principle of Rational Expressions to divide out the common factor from the numerator and denominator.

Example 22.1. Simplify the following rational expression $\dfrac{25a^6b^3}{5a^3b^5}$.

We begin by noting that the GCF of the numerator and denominator is $5a^3b^3$. Using this, we can factor the numerator and denominator to get

$$\frac{25a^6b^3}{5a^3b^5} = \frac{(5a^3b^3)(5a^3)}{(5a^3b^3)(b^2)}.$$

Lastly we can use the Fundamental Principle of Rational Expressions to simplify

$$\frac{(\cancel{5a^3b^3})(5a^3)}{(\cancel{5a^3b^3})(b^2)} = \frac{5a^3}{b^2}.$$

Another way to approach this same problem is to factor both numerator and denominator **completely** and then cancel where appropriate

$$\frac{25a^6b^3}{5a^3b^5} = \frac{\cancel{5} \cdot 5 \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a \cdot \cancel{b} \cdot \cancel{b} \cdot \cancel{b}}{\cancel{5} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{b} \cdot \cancel{b} \cdot \cancel{b} \cdot b \cdot b} = \frac{5a^3}{b^2}.$$

Example 22.2. Simplify:

$$\frac{2x + 2}{x^2 - 1}$$

First we factor both numerator and denominator, then, once in factored form, we can use The Fundamental Principle of Rational Expressions to simplify.

$$\frac{2(x + 1)}{(x - 1)(x + 1)}$$

Now notice that $(x + 1)$ is in common, so, we can cancel it out. Finally, our simplified rational expression is:

$$\frac{2}{(x - 1)}$$

Example 22.3. Simplify:

$$\frac{4x - 16}{x^2 - x - 12} = \frac{4(x - 4)}{(x - 4)(x + 3)} = \frac{4}{(x + 3)}$$

Multiplying Rational Expressions

> **Multiplying Rational Expressions**
> Let P, Q, R, S be polynomials with $Q \neq 0$ and $S \neq 0$
> then
> $$\frac{P}{Q} \cdot \frac{R}{S} = \frac{P \cdot R}{Q \cdot S}$$

Example 22.4. Multiply and write your answer in simplest form:

$$\frac{10x^2}{2y^2} \cdot \frac{14y^5}{5x^3}$$

$$\frac{10x^2}{2y^2} \cdot \frac{14y^5}{5x^3} = \frac{(10x^2)(14y^5)}{(2y^2)(5x^3)} = \frac{140x^2y^5}{10x^3y^2} = \frac{14y^3}{x}.$$

Alternatively, we could do some canceling *before* multiplying and achieve the same result

$$\frac{10x^2}{2y^2} \cdot \frac{14y^5}{5x^3} = \frac{2 \cdot 5 \cdot x^2}{2 \cdot y^2} \cdot \frac{14 \cdot y^2 \cdot y^3}{5 \cdot x^2 \cdot x} = \frac{14y^3}{x}.$$

Example 22.5. Multiply and write the answer in simplest form:

$$\left(\frac{11x^5}{-7y^7z^2}\right)\left(\frac{-10y^5}{33x^3z}\right)\left(\frac{21z^5}{-6x^2}\right)$$

An organized way of doing this problem is to rearrange so that we can handle the "numerical portion" and "variable portion" separately. We are allowed to do this because multiplication of real numbers is commutative, i.e. we can multiply in any order we wish. The rearrangement results in:

$$\left(\frac{11}{-7} \cdot \frac{-10}{33} \cdot \frac{21}{-6}\right) \left(\frac{x^5}{y^7 z^2} \cdot \frac{y^5}{x^3 z} \cdot \frac{z^5}{x^2}\right)$$

Now we can go ahead and do some canceling, carefully, one portion at a time:

$$\frac{\cancel{11}}{-\cancel{7}} \cdot \frac{\cancel{-2} \cdot 5}{\cancel{3} \cdot \cancel{11}} \cdot \frac{\cancel{3} \cdot \cancel{7}}{\cancel{-2} \cdot 3} = -\frac{5}{3}$$

and

$$\frac{\cancel{x^5} \cdot \cancel{x^2}}{y^2 \cdot \cancel{y^5} \cancel{z^2}} \cdot \frac{\cancel{y^5}}{\cancel{x^5} \cdot \cancel{z}} \cdot \frac{\cancel{z^2} \cdot \cancel{z} \cdot z^2}{\cancel{x^2}} = \frac{z^2}{y^2}.$$

Lastly, we put our pieces back together to get the final solution of our problem which is

$$-\frac{5z^2}{3y^2}.$$

Example 22.6. Multiply and simplify:

$$\frac{4}{27x + 18y} \cdot \frac{9x + 6y}{6} = \frac{4}{9(3x + 2y)} \cdot \frac{3(3x + 2y)}{6}$$

Notice that $(3x + 2y)$ is in common in both numerator and denominator and thus can be simplified. So:

$$\frac{4}{27x + 18y} \cdot \frac{9x + 6y}{6} = \frac{4}{9} \cdot \frac{3}{6} = \frac{4 \cdot 3}{9 \cdot 6} = \frac{2}{9}$$

We divide rational expressions in the same way that we divide fractions. We can view the division as the multiplication of the first expression by the *reciprocal* of the second.

Dividing Rational Expressions

> **Dividing Rational Expressions**
> Let P, Q, U, V be polynomials with $Q \neq 0$ and $V \neq 0$
> then
> $$\frac{P}{Q} \div \frac{U}{V} = \frac{P}{Q} \cdot \frac{V}{U} = \frac{P \cdot V}{Q \cdot U}$$

Example 22.7. Divide and write the answer in simplest form:

$$\frac{6e^2 f^3 g}{5e^3 g^3} \div \frac{24 f^6 g^2}{ef}$$

We multiply the first rational expression by the reciprocal of the second:

$$\frac{6e^2 f^3 g}{5e^3 g^3} \cdot \frac{ef}{24 f^6 g^2}$$

Next we can rearrange our problem into numerical and variable portions to get

$$\left(\frac{6}{5} \cdot \frac{1}{24} \right) \left(\frac{e^2 f^3 g}{e^3 g^3} \cdot \frac{ef}{f^6 g^2} \right).$$

Now we independently and carefully simplify each portion

$$\frac{\cancel{6}}{5 \cdot 4 \cdot \cancel{6}} = \frac{1}{20}$$

and

$$\frac{\cancel{e^2} \cdot \cancel{f^3} \cdot \cancel{g}}{\cancel{e^2} \cdot \cancel{e} \cdot \cancel{g} \cdot g^2} \cdot \frac{\cancel{e} \cdot \cancel{f}}{\cancel{f^3} \cdot \cancel{f} \cdot f^2 \cdot g^2} = \frac{1}{f^2 g^4}.$$

Finally we put our pieces back together to form the solution to our problem which is

$$\frac{1}{20 f^2 g^4}.$$

Exit Problem

Simplify: $\dfrac{35x^2b^3}{-21a^2y} \div \dfrac{15x^2b^2}{14ay}$

Chapter 23

Adding and Subtracting Rational Expressions

Think back to the chapter on rational expressions. Let's remember that a **rational expression** (also called an **algebraic fraction**) is one which can be written as a ratio of $\dfrac{P}{Q}$ where P and Q are polynomials and Q $\neq$ 0. We can add and subtract rational expressions in the same way we can add and subtract fractions.

Rational Expressions with "Like" Denominators

Adding and Subtracting Rational Expressions

Let P, Q and R be polynomials with $R \neq 0$

1. $\dfrac{P}{R} + \dfrac{Q}{R} = \dfrac{P+Q}{R}$

2. $\dfrac{P}{R} - \dfrac{Q}{R} = \dfrac{P-Q}{R}$

Example 23.1. Add and write the answer in simplest form:

$$\frac{3y}{16} + \frac{5y}{16}$$

Since the denominators are the same, we just add the numerators to get

$$\frac{3y}{16} + \frac{5y}{16} = \frac{3y + 5y}{16} = \frac{8y}{16} = \frac{y}{2}.$$

Example 23.2. Subtract and write the answer in simplest form:

$$\frac{5x}{x - 3} - \frac{15}{x - 3}$$

We begin by subtracting the numerators to get

$$\frac{5x}{x - 3} - \frac{15}{x - 3} = \frac{5x - 15}{x - 3}.$$

Next, we simplify the rational expression by using a method for we learned in the previous section. We factor the numerator and then cancel the common factor from the numerator and denominator

$$\frac{5x - 15}{x - 3} = \frac{5\cancel{(x - 3)}}{\cancel{(x - 3)}} = 5.$$

Rational Expressions with Unlike Denominators

Recall that in order to combine "unlike" fractions (those with different denominators) we first had to rewrite them with a common denominator. We chose to work with the *least* such common denominator because it streamlined the process. Combining unlike rational expressions requires us to do the same thing. We can find the LCD for rational expressions in exactly the same manner as we found the LCD for fractions, refer to Chapter 2 for the procedure. The only difference to keep in mind is that denominators of rational expressions may be polynomials. But this doesn't hinder us. When we factor our denominators, we simply factor them as products of powers of prime *numbers* and *polynomials*. A **prime (or irreducible) polynomial** is a polynomial which cannot be factored any further.

Example 23.3. Combine $\dfrac{5x}{6} - \dfrac{2x}{3}$.

Recall the steps to find the LCD (Chapter 2):

Step 1. Factor (the denominators): 6 factors into $2 \cdot 3$ and 3 factors into $3 \cdot 1$

Step 2. List (the primes) $2, 3$

Step 3. (Form the) LCD: $2 \cdot 3 = 6$

Now that we have found the LCD, we use it to rewrite each rational expression to transform our problem from

$$\frac{5x}{6} - \frac{2x}{3}$$

to

$$\frac{5x}{6} - \frac{2 \cdot 2x}{2 \cdot 3}$$
$$= \frac{5x}{6} - \frac{4x}{6}.$$

Now we can combine the rational expressions to get

$$\frac{5x}{6} - \frac{4x}{6} = \frac{x}{6}.$$

Example 23.4. Combine $\dfrac{7}{x} + \dfrac{4}{3}$.

We encounter here our first "prime polynomial" which is x. The LCD for this example is $3x$ and so we begin by rewriting the problem to read

$$\frac{7}{x} + \frac{4}{3}$$
$$= \frac{3 \cdot 7}{3 \cdot x} + \frac{4 \cdot x}{3 \cdot x}$$
$$= \frac{21}{3x} + \frac{4x}{3x}.$$

Now, we combine the "like" rational expressions to yield the solution

$$\frac{21 + 4x}{3x}$$

which cannot be simplified any further.

Example 23.5. Combine $\dfrac{5}{4a} - \dfrac{7b}{6}$.

Let us begin by computing the LCD, keeping in mind that here a is a prime polynomial.

Step 1. Factor: $4a$ factors into $2^2 \cdot a$ and 6 factors into $2 \cdot 3$

Step 2. List: $2,\ 3,\ a$

Step 3. LCD: $2^2 \cdot 3 \cdot a = 12a$

Next, we rewrite each fraction using this LCD:

$$\frac{5}{4a} - \frac{7b}{6}$$

$$= \frac{3 \cdot 5}{3 \cdot 4a} - \frac{7b \cdot 2a}{6 \cdot 2a}$$

$$= \frac{15}{12a} - \frac{14ab}{12a}$$

Finally we can combine "like" rational expressions to get

$$\frac{15}{12a} - \frac{14ab}{12a} = \frac{15 - 14ab}{12a}$$

which cannot be reduced any further.

Exit Problem

Simplify: $\dfrac{7}{12} - \dfrac{5}{8b}$

Chapter 24

Solving Fractional Equations

A **fractional equation** is an equation involving fractions which has the unknown in the denominator of one or more of its terms.

Example 24.1. The following are examples of fractional equations:

a) $\dfrac{3}{x} = \dfrac{9}{20}$

b) $\dfrac{x-2}{x+2} = \dfrac{3}{5}$

c) $\dfrac{3}{x-3} = \dfrac{4}{x-5}$

d) $\dfrac{3}{4} - \dfrac{1}{8x} = 0$

e) $\dfrac{x}{6} - \dfrac{2}{3x} = \dfrac{2}{3}$

The Cross-Product property can be used to solve fractional equations.

> **Cross-Product Property**
>
> If $\dfrac{A}{B} = \dfrac{C}{D}$ then $A \cdot D = B \cdot C$.

Using this property we can transform fractional equations into non-fractional ones. We must take care when applying this property and use it only when there is a **single** fraction on each side of the equation. So, fractional equations can be divided into two categories.

I. **Single Fractions on Each Side of the Equation**

   Equations a), b) and c) in Example 24.1 fall into this category. We solve these equations here.

   a) Solve $\dfrac{3}{x} = \dfrac{9}{20}$

Cross-Product	$3 \cdot 20 = 9 \cdot x$
Linear Equation	$60 = 9x$
Divide by 9 both sides	$\dfrac{60}{9} = x$

   The solution is $x = \dfrac{60}{9} = \dfrac{20}{3}$.

   b) $\dfrac{x-2}{x+2} = \dfrac{3}{5}$

Cross-Product	$5 \cdot (x-2) = 3 \cdot (x+2)$
Remove parentheses	$5x - 10 = 3x + 6$
Linear Equation: isolate the variable	$5x - 3x = 10 + 6$

$$2x = 16$$

Divide by 2 both sides
$$\frac{2x}{2} = \frac{16}{2}$$

the solution is $x = 8$.

c) $\dfrac{3}{x-3} = \dfrac{4}{x-5}$

Cross-Product $\qquad\qquad 3 \cdot (x-5) = 4 \cdot (x-3)$

Remove parentheses $\qquad\qquad 3x - 15 = 4x - 12$

Linear Equation: isolate the variable $\qquad 3x - 4x = 15 - 12$

$$-x = 3$$

Divide by -1 both sides $\qquad\qquad \dfrac{-x}{-1} = \dfrac{3}{-1}$

The solution is $x = -3$.

Note: If you have a fractional equation and one of the terms is not a fraction, you can always account for that by putting 1 in the denominator. For example:

Solve
$$\frac{3}{x} = 15$$

We re-write the equation so that all terms are fractions.
$$\frac{3}{x} = \frac{15}{1}$$

Cross-Product	$3 \cdot 1 = 15 \cdot x$
Linear Equation: isolate the variable	$3 = 15x$
Divide by 15 both sides	$\dfrac{3}{15} = \dfrac{15x}{15}$

The solution is $x = \dfrac{3}{15} = \dfrac{3 \cdot 1}{3 \cdot 5} = \dfrac{1}{5}$.

II. **Multiple Fractions on Either Side of the Equation**

Equations d) and e) in Example 24.1 fall into this category. We solve these equations here.

We use the technique for combining rational expressions we learned in Chapter 23 to reduce our problem to a problem with a single fraction on each side of the equation.

d) Solve $\dfrac{3}{4} - \dfrac{1}{8x} = 0$

First we realize that there are two fractions on the LHS of the equation and thus we cannot use the Cross-Product property immediately.
To combine the LHS into a single fraction we do the following:

Find the LCM of the denominators	$8x$
Rewrite each fraction using the LCM	$\dfrac{3 \cdot 2x}{8x} - \dfrac{1}{8x} = 0$
Combine into one fraction	$\dfrac{6x - 1}{8x} = 0$
Re-write the equation so that all terms are fractions	$\dfrac{6x - 1}{8x} = \dfrac{0}{1}$

Cross-Product	$(6x - 1) \cdot 1 = 8x \cdot 0$
Remove parentheses	$6x - 1 = 0$
Linear Equation: isolate the variable	$6x = 1$
Divide by 6 both sides	$\dfrac{6x}{6} = \dfrac{1}{6}$

The solution is $x = \dfrac{1}{6}$.

e) Solve $\dfrac{x}{6} + \dfrac{2}{3x} = \dfrac{2}{3}$

Find the LCM of the denominators of LHS	$6x$
Rewrite each fraction on LHS using their LCM	$\dfrac{x \cdot x}{6x} + \dfrac{2 \cdot 2}{6x} = \dfrac{2}{3}$
Combine into one fraction	$\dfrac{x^2 + 4}{6x} = \dfrac{2}{3}$
Cross-Product	$(x^2 + 4) \cdot 3 = 6x \cdot 2$
Remove parentheses	$3x^2 + 12 = 12x$
Quadratic Equation: Standard form	$3x^2 - 12x + 12 = 0$
Quadratic Equation: Factor	$3 \cdot x^2 - 3 \cdot 4x + 3 \cdot 4 = 0$
	$3(x^2 - 4x + 4) = 0$
	$3(x - 2)(x - 2) = 0$
Divide by 3 both sides	$\dfrac{3(x - 2)(x - 2)}{3} = \dfrac{0}{3}$

$$(x - 2)(x - 2) = 0$$

Quadratic Equation: Zero-Product Property $(x - 2) = 0$ or $(x - 2) = 0$

Since both factors are the same, then $x - 2 = 0$ gives $x = 2$.
The solution is $x = 2$.

Note: There is another method to solve equations that have multiple fractions on either side. It uses the LCM of <u>all</u> denominators in the equation. We demonstrate it here to solve the following equation: $\dfrac{3}{2} - \dfrac{9}{2x} = \dfrac{3}{5}$

Find the LCM of <u>all</u> denominators in the equation $10x$

Multiply every fraction (both LHS and RHS) by the LCM $10x \cdot \dfrac{3}{2} - 10x \cdot \dfrac{9}{2x} = 10x \cdot \dfrac{3}{5}$

$$\dfrac{10x \cdot 3}{2} - \dfrac{10x \cdot 9}{2x} = \dfrac{10x \cdot 3}{5}$$

Simplify every fraction $\dfrac{5x \cdot 3}{1} - \dfrac{5 \cdot 9}{1} = \dfrac{2x \cdot 3}{1}$

See how all denominatiors are now 1, thus can be disregarded $5x \cdot 3 - 5 \cdot 9 = 2x \cdot 3$

Solve like you would any other equation $15x - 45 = 6x$

Linear equation: islolate the variable $15x - 6x = 45$

$$9x = 45$$

$$x = \dfrac{45}{9}$$

$$x = 5$$

The solution is $x = 5$.

Exit Problem

Solve: $\dfrac{2}{x} + \dfrac{1}{3} = \dfrac{1}{2}$

Chapter 25

Rectangular Coordinate System

In Chapter 1, we learned that a real number can be represented as a point on a number line. In this section we will learn that an ordered pair of real numbers can be represented as a point in a plane. This plane is called a rectangular coordinate system or Cartesian coordinate system named after a French mathematician, René Descartes.

The figure on the next page shows the rectangular coordinate system. It consists of two number lines perpendicular to each other. The horizontal line is called x-axis, and the vertical line is called y-axis. The two number lines intersect at the origin of each number line and the intersection is called the origin of the coordinate system. The axes divide the plane into four regions. They are called quadrants. From the top right to the bottom right, the quadrants are ordered as I, II, III and IV in a counter-clockwise direction.

Every point P on the plane is represented by an ordered pair (x, y), which is referred to as the coordinates of the point P. The first value is called x-coordinate, and the second one is called y-coordinate. The x-coordinate measures the distance of the point from the y-axis, and the y-coordinate measures the distance of the point from the x-axis. If $x > 0$, the point is to the right of the y-axis; if $x < 0$, the point is to the left of the y-axis; if $x = 0$, the point is on the $y-$axis. If $y > 0$, the point is above the x-axis; if $y < 0$, the point is below the y-axis; if $y = 0$, the point is on the x-axis.
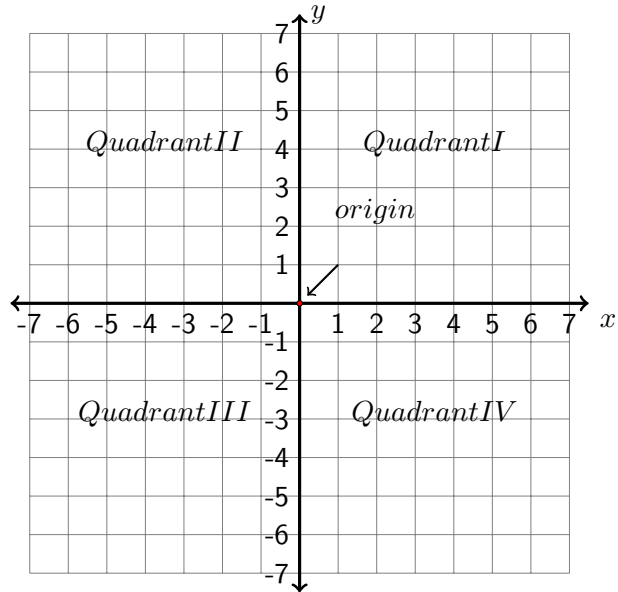
**Example 25.1.** Plot the given points on the rectangular coordinate system.

(A) $(4, 2)$,　(B) $(-2, 3)$,　(C) $(-3, -1)$,　(D) $(2, -4)$

(E) $(0, 1)$,　(F) $(-1, 0)$
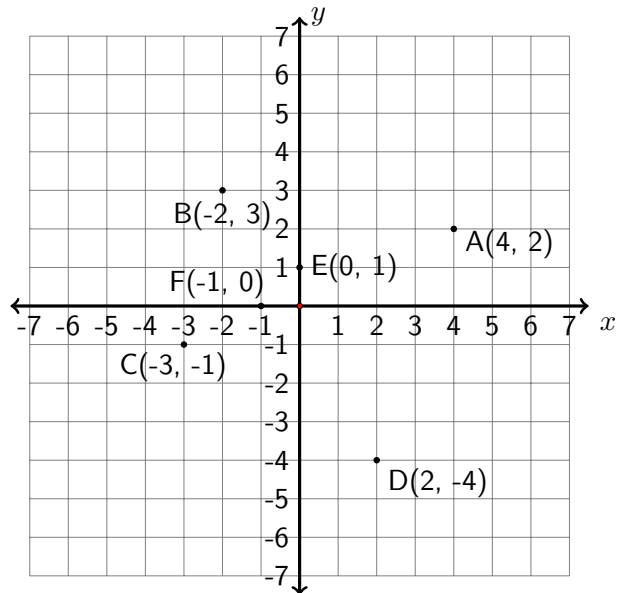
To plot point (4,2), we start from the origin then move 4 units to the right and 2 units up. The point (4,2) is labeled as A. To plot point (-2,3), we start from the origin then move 2 units to the left and 3 units up. The point (-2,3) is labeled as B. Points (C) and (D) are plotted in a similar way. To plot (0,1), from the origin, we only move 1 unit up, so the point E is on the y-axis. Similarly, the point (-1,0) is on the x-axis, 1 unit to the left of the origin. The point (0,0) is just the origin of the coordinate system.

Example 25.2. Determine the coordinates of following points:
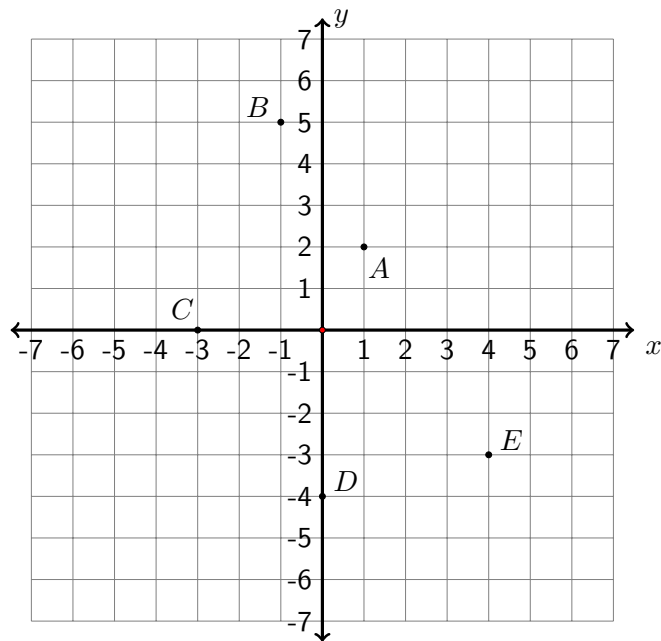
Point A	1 unit right, 2 units up	$(1,2)$
Point B	1 unit left, 5 units up	$(-1,5)$
Point C	3 units left, 0 units up or down	$(-3,0)$
Point D	0 units left or right, 4 units down	$(0,-4)$
Point E	4 units right, 3 units down	$(4,-3)$

Linear Equations in Two Variables

In Chapter 16, we solved linear equations in one variable. In this section, we will learn how to find solutions of linear equations in two variables, and how to represent them in a rectangular coordinate system.

The equation $y = 2x + 3$ is an example of a linear equation in two variables, x and y. The solution to this equation is an ordered pair of two real numbers (a, b) so that when x is replaced by a and y is replaced by b, the equation is a true statement. For example, $(1,5)$ is a solution of $y = 2x + 3$ because when we substitute 1 for x and 5 for y, we get $5 = 2(1) + 3$, which is a true statement. The ordered pair $(4,3)$ is not a solution to the equation because substituting 4 for x and 3 for y does not satisfy the equation i.e. $3 \neq 2(4) + 3$.

Example 25.3. Check if the following ordered pairs are solutions of the equation $2x - 3y = 5$.

$$\text{(A) } (1, -1) \quad \text{(B) } (2, 1) \quad \text{(C) } \left(\frac{5}{2}, 0\right)$$

To check, let's substitute the values into our equation.

(A) $\quad 2(1) - 3(-1) = 5 \quad \implies \quad 2 + 3 = 5 \quad \implies \quad 5 = 5 \quad$ True

(B) $\quad 2(2) - 3(1) = 5 \quad \implies \quad 4 - 3 = 5 \quad \implies \quad 1 = 5 \quad$ False

(C) $\quad 2\left(\frac{5}{2}\right) - 3(0) = 5 \quad \implies \quad 5 - 0 = 5 \quad \implies \quad 5 = 5 \quad$ True

Therefore (A) and (C) are solutions and (B) is not a solution.

From the last example we see that the solutions of an equation in two variables are ordered pairs (x, y). Therefore they can be represented by points on a rectangular coordinate system. By connecting all possible points, we get the *graph* of the equation.

Now let's find the graph of the equation $y = 2x + 3$ by finding solution pairs and plotting them. Since we have only one equation for two variables x and y, we can first choose a few x values, say $x = -2, -1, 0, 1, 2$, and then find their corresponding y values. For example, if $x = -2$ then $y = 2(-2) + 3 = -1$. So $(-2, -1)$ is a solution pair. It is helpful to set up a table to keep track.

Value of x	Calculate $y = 2x + 3$	Solution Pair
-2	$y = 2(-2) + 3 = -1$	$(-2, -1)$
-1	$y = 2(-1) + 3 = 1$	$(-1, 1)$
0	$y = 2(0) + 3 = 3$	$(0, 3)$
1	$y = 2(1) + 3 = 5$	$(1, 5)$
2	$y = 2(2) + 3 = 7$	$(2, 7)$
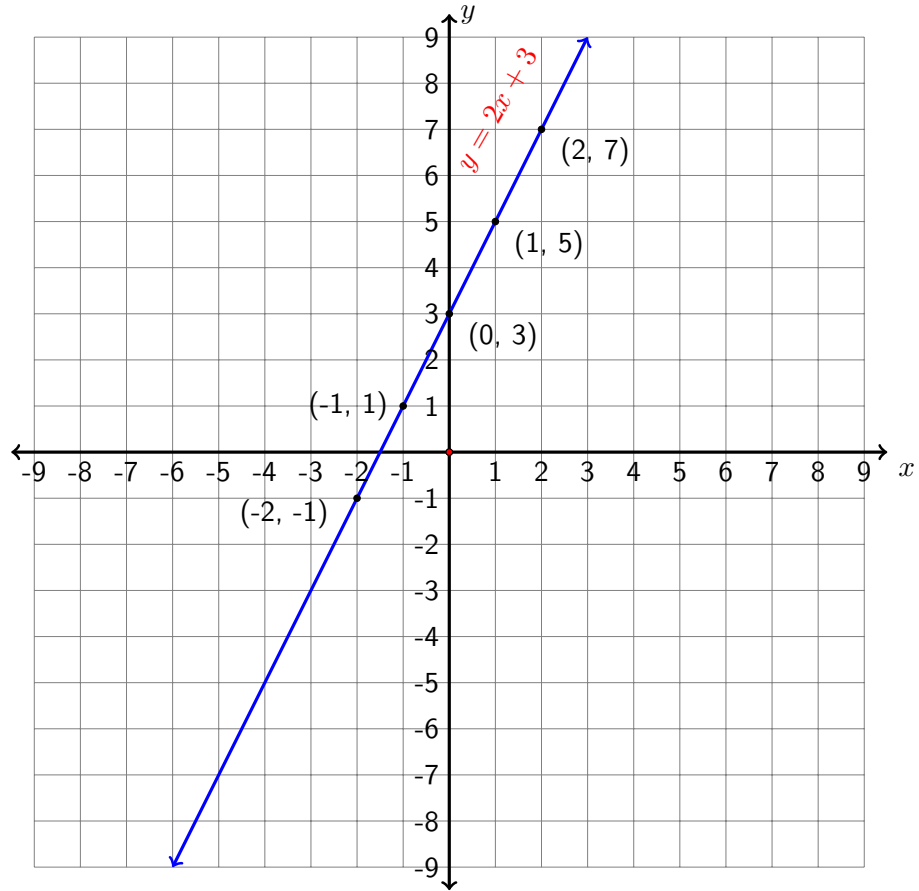
After plotting these points on a rectangular coordinate system, it is easy to guess that the graph of $y = 2x + 3$ might be a straight line. In fact, the graph of the equation $y = 2x + 3$ *is* a straight line - which is why it is referred to as a *linear* equation.

Example 25.4. Graph the equation $3x + 2y = 4$ by plotting points.
To graph the equation, we need to first find a couple of solution pairs. Let's begin by solving for y:
$$3x + 2y = 4 \quad \implies \quad 2y = -3x + 4 \quad \implies \quad y = \frac{-3x+4}{2} \quad \implies \quad y = \frac{-3x}{2} + 2$$
Let $x = -4, -2, 0, 2, 4$. Here only even numbers are chosen so the results are not fractions.

Value of x	Calculate $y = \dfrac{-3x}{2} + 2$	Solution Pair
-4	$y = \dfrac{-3(-4)}{2} + 2 = 8$	$(-4, 8)$
-2	$y = \dfrac{-3(-2)}{2} + 2 = 5$	$(-2, 5)$
0	$y = \dfrac{-3(0)}{2} + 2 = 2$	$(0, 2)$
2	$y = \dfrac{-3(2)}{2} + 2 = -1$	$(2, -1)$
4	$y = \dfrac{-3(4)}{2} + 2 = -4$	$(4, -4)$

The graph of the line $3x + 2y = 4$ is shown in the following figure.
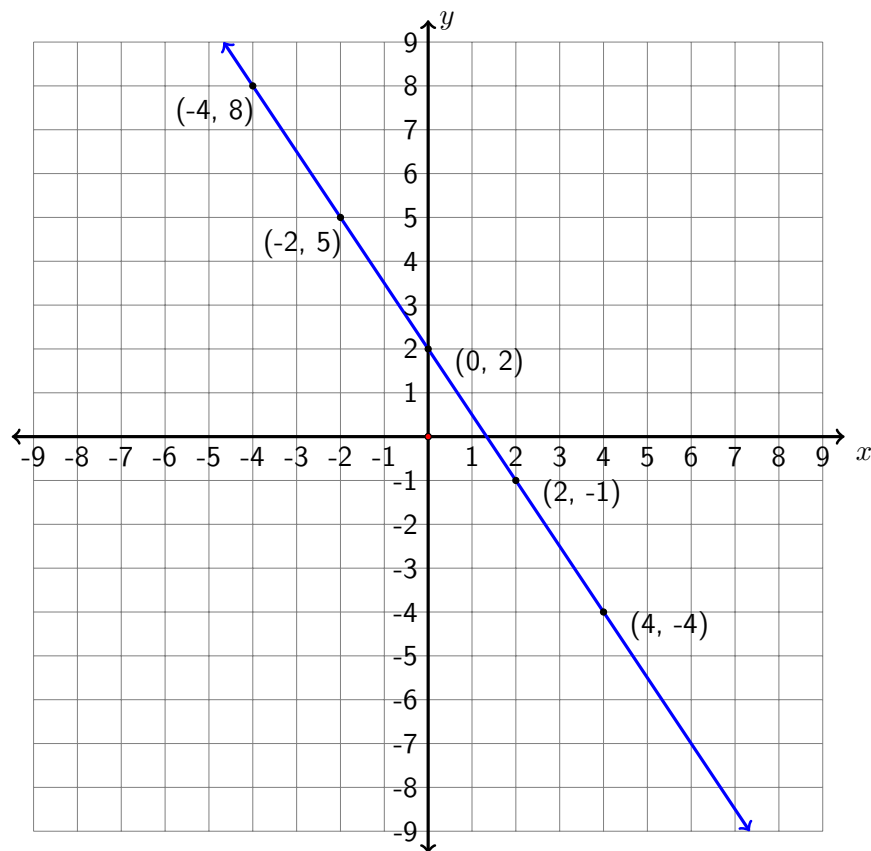
Exit Problem

Graph the equation: $3x + 2y = 8$

Chapter 26

Graphing Linear Equations

In general, a linear equation in two variables is of the form

$$Ax + By = C,$$

in which A, B, C are real numbers and A and B cannot both be 0. This is called the **standard form of a linear equation**. Here are a few examples of linear equations in standard form:

$$3x + 2y = 4, \quad -0.25x + \frac{2}{3}y = 1, \quad -3x = 2, \quad 4y = 3.$$

Some linear equations are not written in standard form, but they can be easily transformed to a standard form. For example, the linear equation $y = -2x - 3$ can be written in a standard form of $2x + y = -3$ by adding $2x$ to both sides of the equation.

The graph of a linear equation is a straight line in the rectangular coordinate system. In the last chapter, we learned how to plot a straight line by plotting some points of a given linear equation. In this chapter, we will learn some important concepts of linear equations and use them to plot straight lines.

Graphing Linear Equations Using Intercepts

On the graph of a straight line in the rectangular coordinate system, there are two special points, the x-intercept and the y-intercept.

From geometry, we know that a straight line is fully determined by two points. Therefore, after identifying the intercepts of a given linear equation, we can plot its graph by using these two points. It is a good practice to find a

155

third point. If the third point doesn't lie on the straight line determined by the intercepts, this indicates that we may have made an error in our calculations.

The x-intercept

The x-intercept is the point where the graph intersects the
x-axis.
To find the x-intercept, evaluate the equation at $y = 0$.
The x-intercept will look like $(x\text{-intercept}, 0)$.

The y-intercept

The y-intercept is the point where the graph intersects
y-axis.
To find the y-intercept, evaluate the equation at $x = 0$.
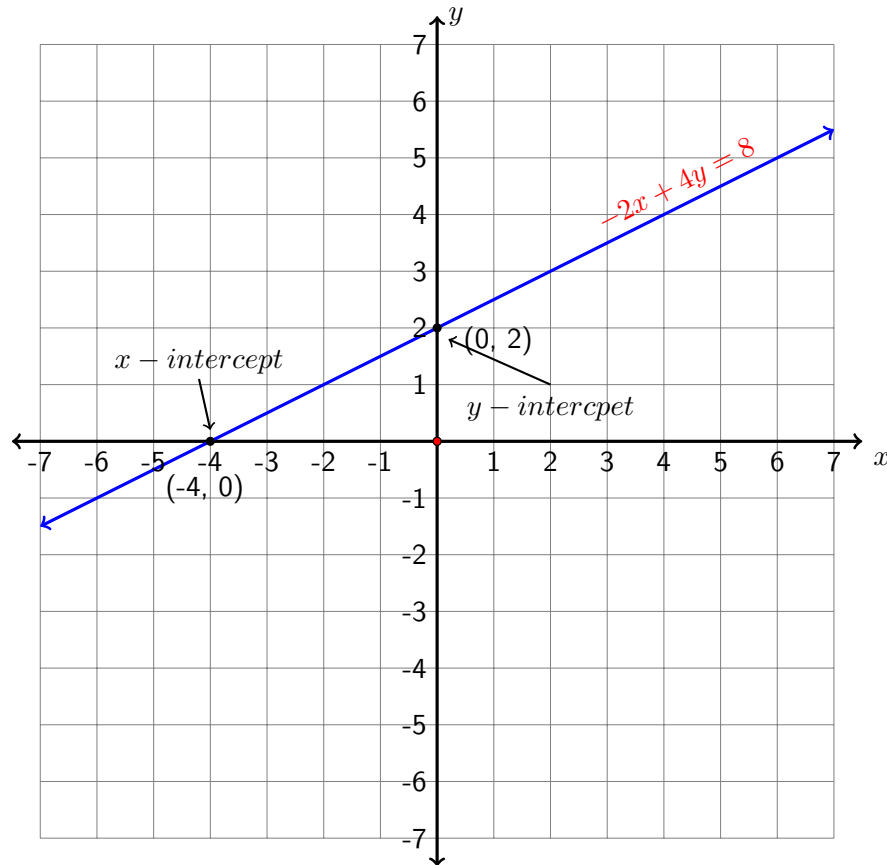The y-intercept will look like $(0, y\text{-intercept})$.

Example 26.1. Find the intercepts of $-2x + 4y = 8$ and use them to graph the linear equation.

$$\text{Let } y = 0 \quad \implies \quad -2x + 4(0) = 8 \quad \implies \quad -2x = 8 \quad \implies \quad x = -4$$

Therefore, the $x-$intercept is $(-4, 0)$.

$$\text{Let } x = 0 \quad \implies \quad -2(0) + 4y = 8 \quad \implies \quad 4y = 8 \quad \implies \quad y = 2$$
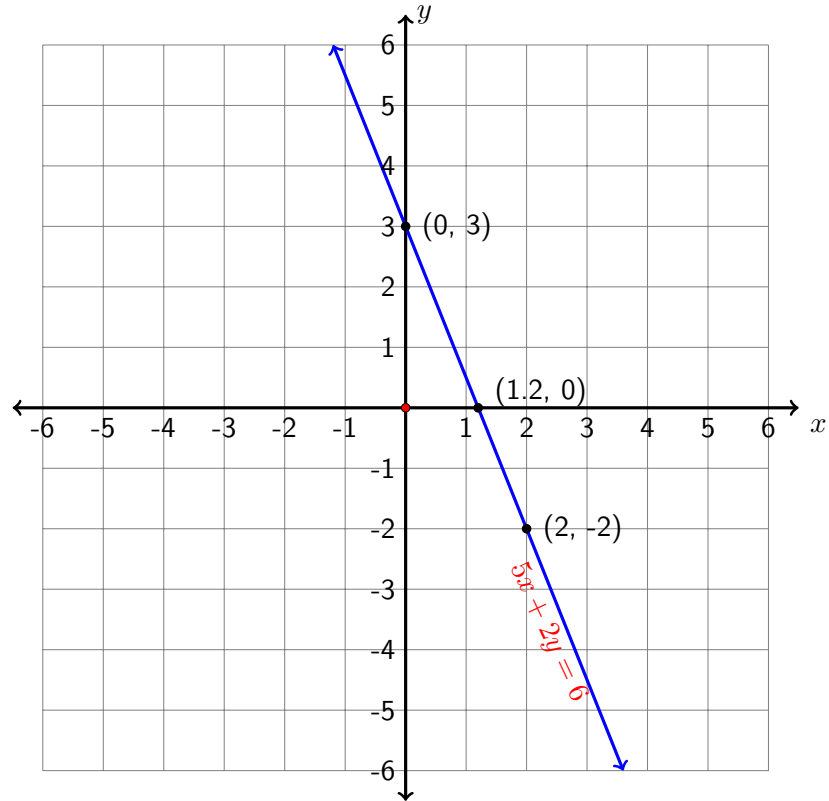
Therefore, the $y-$intercept is $(0, 2)$

Example 26.2. Find the intercepts of $5x + 2y = 6$ and use them to graph the equation. We first find the two intercepts (and then find a third point).

$$\text{Let } y = 0 \quad \Longrightarrow \quad 5x + 2(0) = 6 \quad \Longrightarrow \quad 5x = 6 \quad \Longrightarrow \quad x = \frac{6}{5} = 1.2$$

$$\text{Let } x = 0 \quad \Longrightarrow \quad 5(0) + 2y = 6 \quad \Longrightarrow \quad 2y = 6 \quad \Longrightarrow \quad y = 3$$

$$\text{Let } x = 2 \quad \Longrightarrow \quad 5(2) + 2y = 6 \quad \Longrightarrow \quad 10 + 2y = 6 \quad \Longrightarrow \quad y = -2$$

Now we have three solution pairs: (1.2, 0), (0, 3) and (2, -2). The following figure is the graph of the equation $5x + 2y = 6$.

The Slope of a Line

Different straight lines have different steepness. Mathematically, there is a number that can measure the steepness of a given line. This number is called the *slope* of a line, and is denoted by the letter m. The following figure helps us to determine the formula for calculating the slope m.

In the figure below, P and Q are two arbitrary points on the given line. Point P has the coordinates (x_1, y_1), and point Q has the coordinates (x_2, y_2). Here we use subscripts to distinguish coordinates for different points. The slope m is defined as the ratio of the change of the two points in the y direction to the change of the two points in the x direction.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$
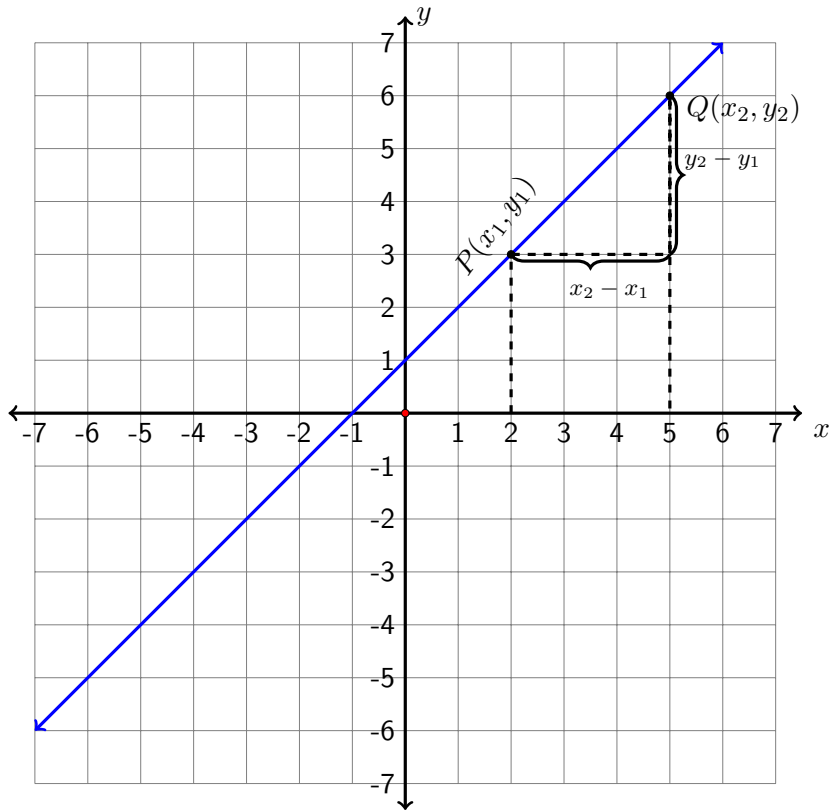
Since the y direction is vertical, the change in y coordinates referred to as the "rise." Similarly, the x direction is horizontal so the change in x coordinates is referred to as the "run." So another way of remembering the formula for finding the slope is to remember that it is the "rise over run."

> **The Slope of a Line**
> The slope of a line passing through (x_1, y_1) and (x_2, y_2) is
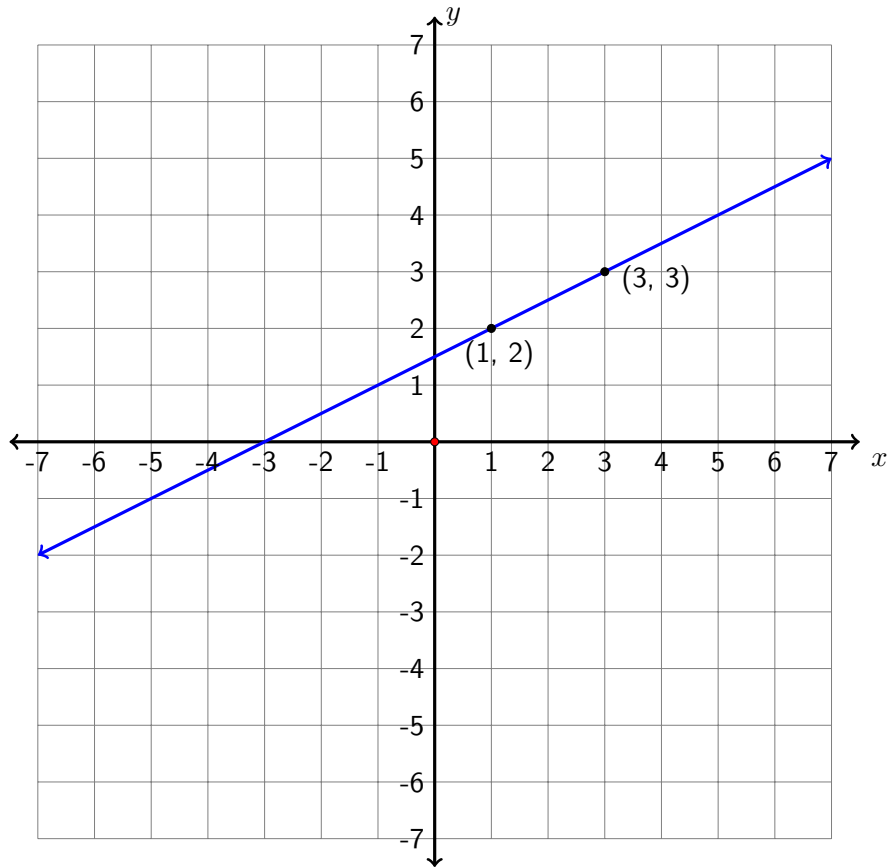> $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

Example 26.3. Find the slope of the line passing through the following points and use the given points to graph the line.

a) $(1, 2)$ and $(3, 3)$

We choose $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (3, 3)$. Actually it doesn't matter which point we assign as the first and the second point. We will always get the same slope.

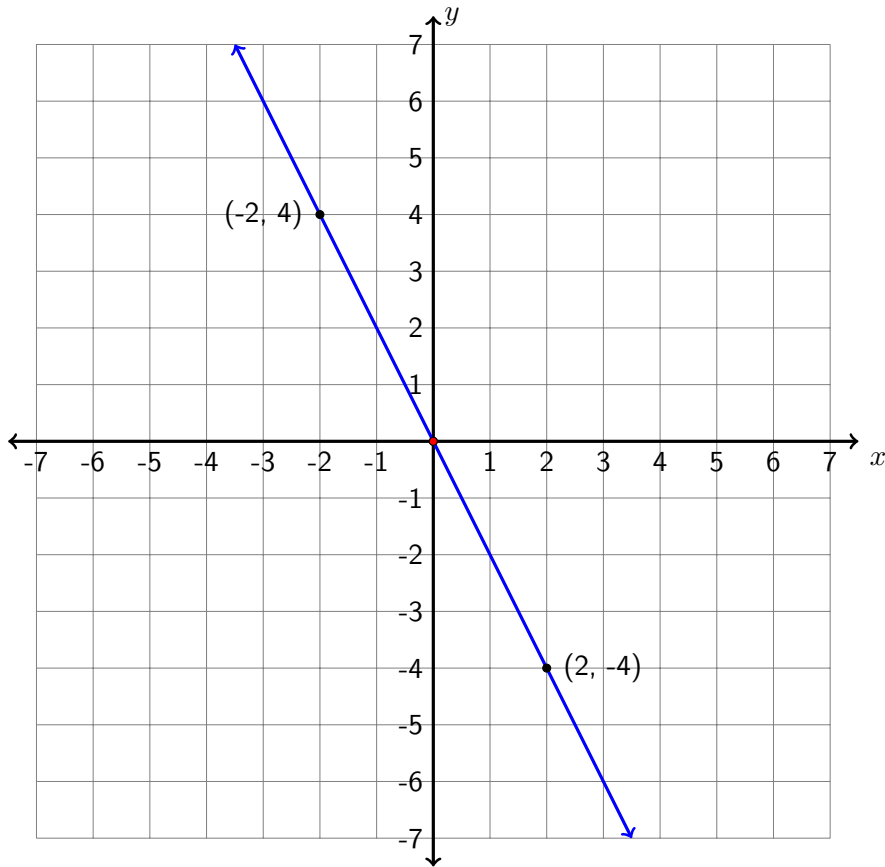$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 2}{3 - 1} = \frac{1}{2}.$$

The slope is positive. The graph shows it is a rising line from left to right.

b) $(-2, 4)$ and $(2, -4)$

Let $(x_1, y_1) = (-2, 4)$ and $(x_2, y_2) = (2, -4)$. Then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 4}{2 - (-2)} = \frac{-8}{4} = -2.$$

The slope is negative. The graph shows it is a falling line from left to right, and this line is steeper than the line in part a).
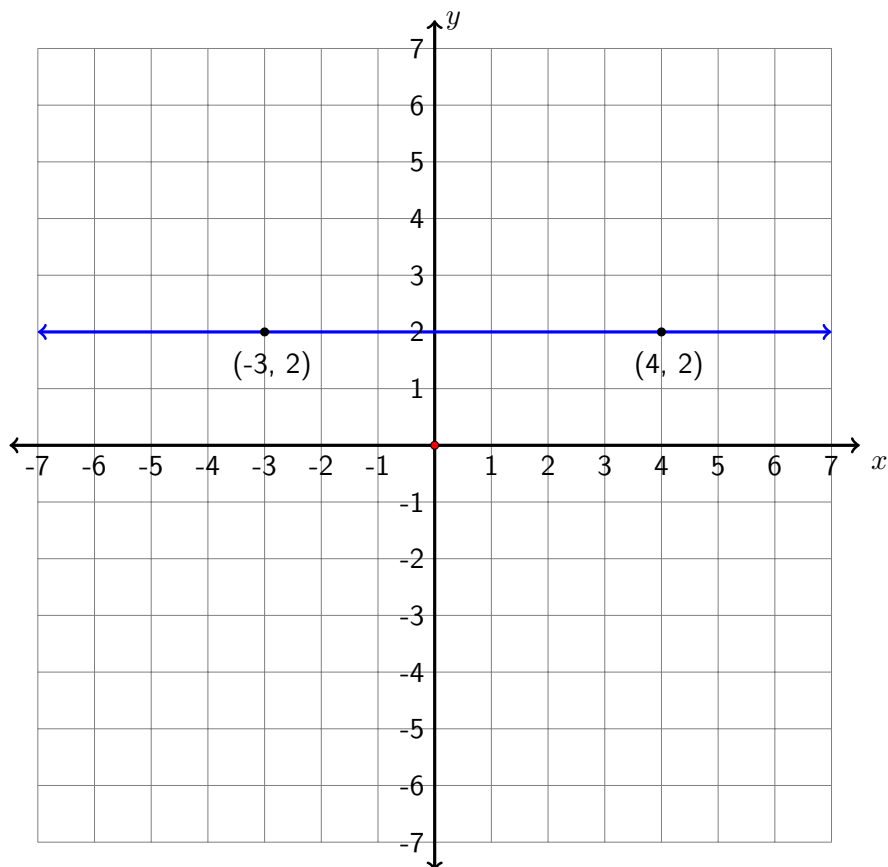
c) $(-3, 2)$ and $(4, 2)$

Let $(x_1, y_1) = (-3, 2)$ and $(x_2, y_2) = (4, 2)$. Then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 2}{4 - (-3)} = \frac{0}{7} = 0.$$

The slope is 0. The graph shows it is a horizontal line.

Note: All horizontal lines are characterized by points that have the same y value, so they all have a slope 0.
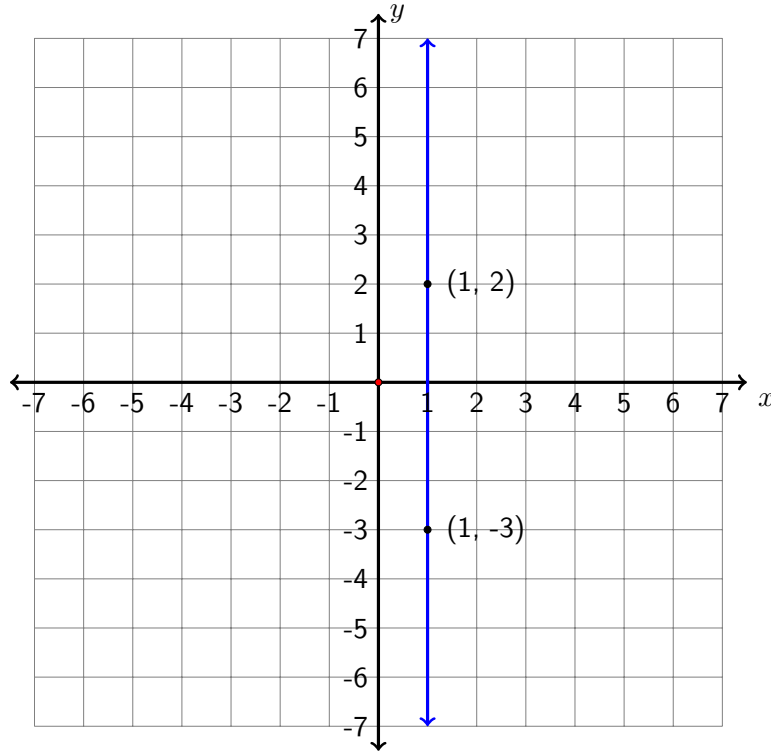
d) $(1, 2)$ and $(1, -3)$

Let $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (1, -3)$. Then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 2}{1 - 1} = \frac{-5}{0} = \text{undefined}$$

The slope is undefined. The graph shows it is a vertical line.

Note: All vertical lines are characterized by points that have the same x value, so they all have an undefined slope.

Properties of the Slope

1. If the slope is positive, the line goes upward from left to right.

2. If the slope is negative, the lines goes downward from left to right.

3. If the slope is 0, the line is horizontal, that is, there is no rise in the y direction.

4. If the slope is undefined, the line is vertical, that is, there is no run in the x direction.

5. The larger the absolute value of the slope, the steeper the straight line.

Slope-Intercept Form of the Equation of a Line $y = mx + b$

In the previous section, learned that to find the slope of a line, we need two points on the line. Given an equation of a line, we can easily find two points on the line by finding two solution pairs of the equation. However, there is another way to identify the slope.

Let's find the slope of the line given by the equation: $2x - 3y = 4$. We can choose *any* two points on the line, so let's choose to work with the x and y-intercepts.

$$\text{Let } x = 0 \quad \implies \quad 2(0) - 3y = 4 \quad \implies \quad -3y = 4 \quad \implies \quad y = -\frac{4}{3}$$

$$\text{Let } y = 0 \quad \implies \quad 2x - 3(0) = 4 \quad \implies \quad 2x = 4 \quad \implies \quad x = 2$$

Therefore the intercepts are: $(0, -\frac{4}{3})$ and $(2, 0)$ and so the slope is :

$$m = \frac{0 - (-\frac{4}{3})}{2 - 0} = \frac{\frac{4}{3}}{2} = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3}$$

Suppose we return to our original equation and solve it for y:

$$2x - 3y = 4 \quad \implies \quad -3y = -2x + 4$$

$$\implies \quad y = \frac{-2x + 4}{-3}$$

$$\implies \quad y = \frac{2}{3}x - \frac{4}{3}$$

Rewriting the equation in this way helps us to identify two numerical values $\frac{2}{3}$ and $-\frac{4}{3}$. The first value is exactly the same as the slope and the second is exactly the same as the y-intercept! This is true in general. If the equation of a

line is written in the form $y = mx + b$, then the coefficient of the x term is the slope of the line and the constant term is the y-coordinate of the $y-$intercept (so we write $(0, b)$).

> **The slope-intercept form of the equation of a line**
>
> $$y = mx + b$$
>
> is called the slope-intercept form of the equation of a line
> with slope m and y-interecpt $(0, b)$.

Example 26.4. Find the slope and $y-$intercept of the line by rewriting the line equation $x + 4y = 8$ in the slope-intercept form.

$$x + 4y = 8 \quad \Longrightarrow \quad 4y = -x + 8$$

$$\Longrightarrow \quad y = \frac{-x + 8}{4}$$

$$\Longrightarrow \quad y = -\frac{1}{4}x + 2$$

The slope is $-\frac{1}{4}$ and the $y-$intercept is $(0, 2)$. Reminder: The slope is a **numerical value**. Don't let the x variable tag along when you write it down!

Graphing Linear Equations Using the Slope and the y-intercept

Using the $y-$intercept and the slope of a line, we can easily sketch the graph of a straight line. Remember that we only need two points to graph a line. We use the $y-$intercept as the first point, then we use the slope and the $y-$intercept to get another point.

Example 26.5. Graph the given linear equation by using the slope and the y-intercept:
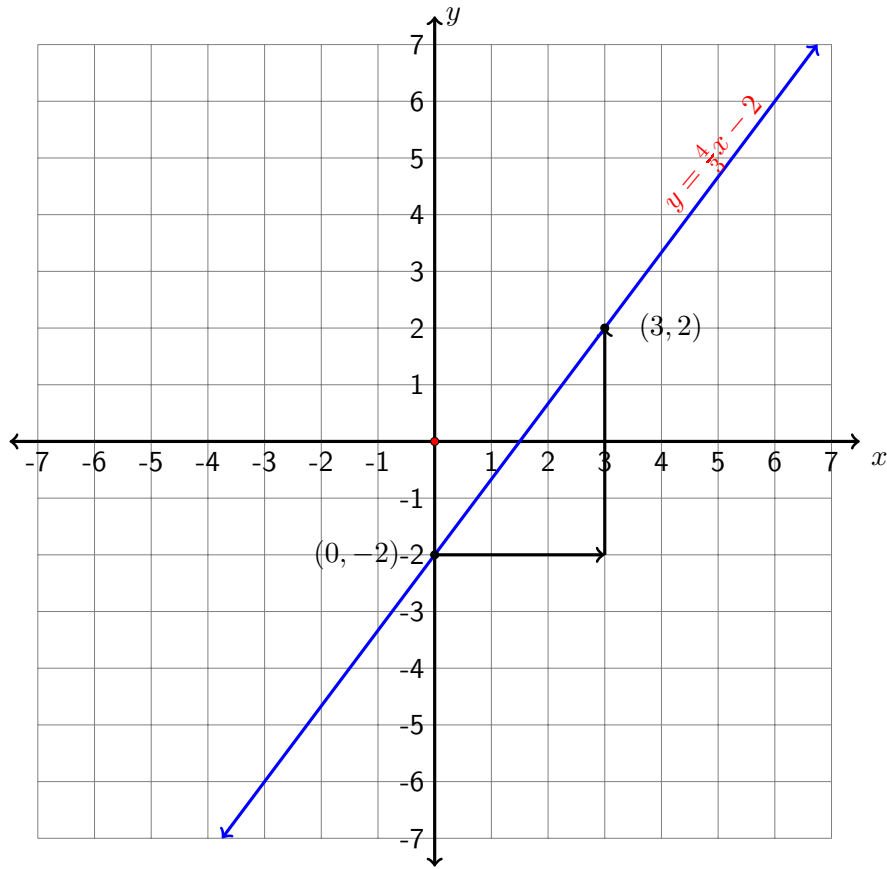
a) $y = \frac{4}{3}x - 2$

Step 1 Write the linear equation in the slope-intercept form. In this problem the equation given is already in $y = mx + b$ form, so we can identify that the slope is $m = \frac{4}{3}$ and the $y-$intercept is $(0, -2)$.

Step 2 Plot the $y-$intercept $(0, -2)$ on the $y-$axis of the coordinate system.

Step 3 Use the slope $m = \frac{4}{3} = \frac{\text{rise}}{\text{run}}$ to locate the second point, that is, from the $y-$intercept we move to the right (run) three units and up (rise) 4 units to plot a second point at $(3, 2)$.

Step 4 Draw a line passing though the two points.

b) $x + 4y = 8$

Step 1 Write the linear equation in the slope-intercept form. (Refer to Example 26.4 to see how this is done):

$$x + 4y = 8 \quad \Longrightarrow \quad y = -\frac{1}{4}x + 2$$

Step 2 Plot the y-intercept $(0, 2)$ on the y-axis of the coordinate system.
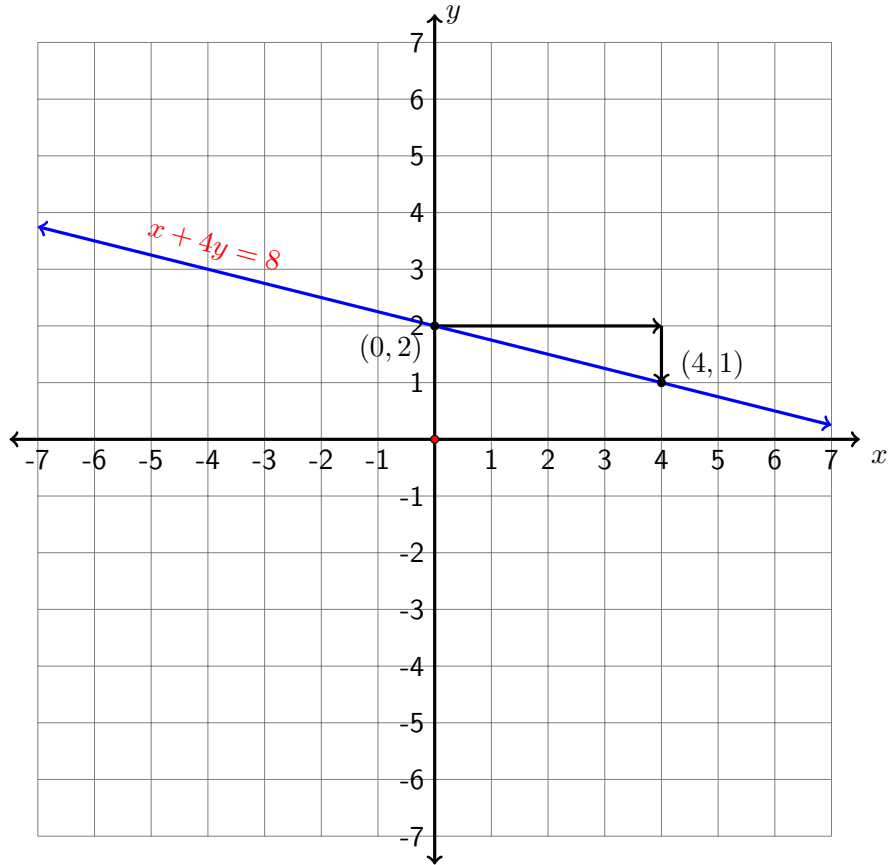
Step 3 Using the slope $m = -\frac{1}{4} = \frac{-1}{4} = \frac{\text{rise}}{\text{run}}$ to locate the second point, that is, from the y-intercept we move to the right (run) four units and down (rise) one unit (because the change in y is negative, we move down) and plot a second point at $(4, 1)$.

Here we chose to put the negative sign in the numerator, but we could have easily assigned the negative sign to the denominator. This can be done because

$$-\frac{1}{4} = \frac{-1}{4} = \frac{1}{-4}.$$

If we decided to work with the slope written as $m = \dfrac{1}{-4}$, from the y-intercept $(0, 2)$ we would move four units *left* (run) and one unit *up* (rise). This would land us at the point $(-4, 3)$.

Step 4 Draw a line passing though the two points. Whichever two points we end up working with either $(0, 2)$ and $(4, 1)$ or $(0, 2)$ and $(-4, 3)$ - we get the same line, try it! The figure below shows the line as drawn from the points $(0, 2)$ and $(4, 1)$.

c) $2x - y = 0$

Step 1 Write the linear equation in the slope-intercept form.

$$2x - y = 0 \quad \implies \quad -y = -2x \quad \implies \quad y = 2x$$

Here the slope is 2 and the $y-$intercept is $(0,0)$.

Step 2 Plot the $y-$intercept $(0,0)$ on the $y-$axis, which is the origin of the coordinate system.

Step 3 Using the slope $m = 2 = \frac{2}{1} = \frac{\text{rise}}{\text{run}}$ to locate the second point, that is, from the $y-$intercept we move to the right (run) one unit and up (rise) two units, plot a second point at $(1,2)$.

Step 4 Draw a line passing through the two points.
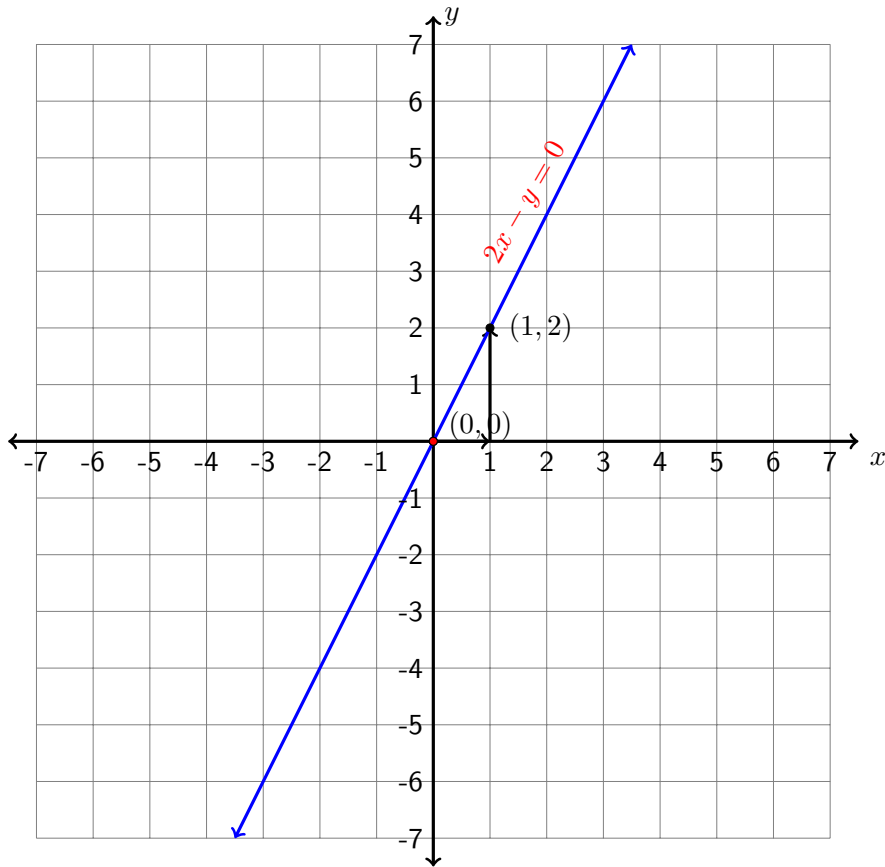
Point-Slope Form of the Equation of a Line $y - y_0 = m(x - x_0)$

Suppose we are given only the slope of a line and one point on the line and are asked to find its equation. Even though we do not know the y-intercept offhand, we can still find the equation of the line by using what's called the *point-slope* formula (or form).

<div style="border: 2px solid orange; background: #fdf6c3; border-radius: 10px; padding: 10px;">

The point-slope form of the equation of a line

$$y - y_0 = m(x - x_0)$$

is called the point-slope form of the equation of a line of
slope m passing through the point (x_0, y_0).

</div>

Example 26.6. Find the equation of the line passing through the point $(1, -2)$
with slope $\dfrac{1}{5}$.

We use the point-slope formula here, with $m = \dfrac{1}{5}$, $x_0 = 1$ and $y_0 = -2$.

So, the equation of the line will be given by $y - (-2) = \dfrac{1}{5}(x - 1)$.

$y + 2 = \dfrac{1}{5}x - \dfrac{1}{5}$, so $y = \dfrac{1}{5}x - \dfrac{1}{5} - 2$, and

$$y = \frac{1}{5}x - \frac{11}{5}.$$

Note: The equation $y = \dfrac{1}{5}x - \dfrac{11}{5}$ is written in the slope-intercept form,
therefore we can read off the y-intercept if we wish: $\left(0, -\dfrac{11}{5}\right)$.

Example 26.7. Find the equation of the line passing through the points $(3, 2)$
and $(-4, 3)$

Notice that here we are *not* given the slope. We need to find it before we
can proceed to find the equation of the line.

The slope is $m = \dfrac{3 - 2}{-4 - 3} = -\dfrac{1}{7}$.

Now, we can choose any of the given points as our one point and use the
point-slope form to find the equation of the line. Let's use $(3, 2)$.

So, with $m = -\dfrac{1}{7}$, $x_0 = 3$ and $y_0 = 2$, the equation of the line will be given by

$$y - 2 = -\frac{1}{7}(x - 3)$$

$$y - 2 = -\frac{1}{7}x + \frac{3}{7} = -\frac{1}{7}x + \frac{3}{7} + 2$$

and, finally,

$$y = -\frac{1}{7}x + \frac{17}{7}.$$

Note: Again, if we wish to read off the $y-$intercept, it is $(0, \frac{17}{7})$.

Equations of Horizontal and Vertical Lines

Recall that all horizontal lines have slope 0.

> **Equation of a Horizontal Line**
> The equation of a horizontal line passing through the point
> (a, b) is $y = b$.

Recall that all vertical lines have an undefined slope.

> **Equation of a Vertical Line**
> The equation of a vertical line passing through the point
> (a, b) is $x = a$.

Example 26.8. a) The equation of the horizontal line passing through the point $(3, -7)$ is $y = -7$.

b) The equation of the vertical line passing through the point $(\frac{2}{5}, -1)$ is $x = \frac{2}{5}$.

c) The equation of the horizontal line passing through the point $(4, 0)$ is $y = 0$.

d) The equation of the vertical line passing through the point $(-\frac{1}{2}, 0)$ is $x = -\frac{1}{2}$.

> **Exit Problem**
>
> Indicate the slope and the y-intercept, and graph the
> equation: $6x + 4y = 6$

Chapter 27

Solving a System of Linear Equations Algebraically

Suppose that Adam has 7 bills, all fives and tens, and that their total value is $40. How many of each bill does he have? In order to solve such a problem we must first define variables.

Let x be the number of five dollar bills.
Let y be the number of ten dollar bills.

Next, we write equations that describe the situation:

$$\begin{array}{rcll} x & + & y & = & 7 & : \text{Adam has 7 bills.} \\ 5x & + & 10y & = & 40 & : \text{The combined value of the bills is } \$40. \end{array}$$

That is, we must solve the following system of two linear equations in two variables (unknowns):

$$\left(\begin{array}{rclcl} x & + & y & = & 7 \\ 5x & + & 10y & = & 40 \end{array} \right)$$

This chapter deals with solving systems of two linear equations with two variable, such as the one above. We will consider two different algebraic methods: the substitution method and the elimination method.

The Substitution Method

In this section we solve systems of two linear equations in two variables using the **substitution method**. To illustrate, we will solve the system above with

172

this method. We begin by solving the first equation for one variable in terms of the other. In this case we will solve for the variable y in terms of x:

$$x + y = 7$$
$$\implies y = 7 - x$$

Next, we **substitute** $y = 7 - x$ into the second equation $5x + 10y = 40$:

$$5x \;+\; 10(7 - x) \;=\; 40$$

The equation above can now be solved for x since it only involves one variable:

$$
\begin{aligned}
5x + 10(7 - x) &= 40 & \\
5x + 70 - 10x &= 40 & \text{distribute } 10 \text{ into the parentheses} \\
-5x + 70 &= 40 & \text{collect like terms} \\
-5x &= -30 & \text{subtract } 70 \text{ from both sides} \\
x &= 6 & \text{divide both sides by } -5
\end{aligned}
$$

Hence, we get $x = 6$. To find y, we substitute $x = 6$ into the first equation of the system and solve for y (Note: We may substitute $x = 6$ into either of the two original equations or the equation $y = 7 - x$):

$$
\begin{aligned}
6 + y &= 7 & \\
y &= 1 & \text{subtract } 6 \text{ from both sides}
\end{aligned}
$$

Therefore the solution to the system of linear equations is

$$x = 6, \quad y = 1.$$

Before we are truly finished, we should **check** our solution. The solution of a system of equations are the values of its variables which, when substituted into the two original equations, give us true statements. So to check, we substitute $x = 6$ and $y = 1$ into each equation of the system:

$$x + y = 7 \implies 6 + 1 = 7 \implies 7 = 7 \;\; \text{true!}$$
$$5x + 10y = 40 \implies 5(6) + 10(1) = 40 \implies 30 + 10 = 40 \implies 40 = 40 \;\; \text{true!}$$

Hence, our solution is correct. To answer the original word problem – recalling that x is the number of five dollar bills and y is the number of ten dollar bills – we have that:

Adam has 6 five dollar bills and 1 ten dollar bill.

Example 27.1.

$$\left(\begin{array}{ccccc} 6x & + & 2y & = & 72 \\ 3x & + & 8y & = & 78 \end{array} \right)$$

Again, here we solve the system of equations using substitution. First, solve the first equation $6x + 2y = 72$ for y:

$$\begin{aligned} 6x + 2y &= 72 \\ \implies \quad 2y &= -6x + 72 \quad & \text{subtract } 6x \text{ from both sides} \\ \implies \quad y &= -3x + 36 \quad & \text{divide both sides by } 2 \end{aligned}$$

Substitute $y = -3x + 36$ into the second equation $3x + 8y = 78$:

$$\begin{aligned} 3x + 8y &= 78 \\ \implies \quad 3x + 8(-3x + 36) &= 78 \\ \implies \quad x &= 10 \end{aligned}$$

Hence $x = 10$. Now substituting $x = 10$ into the equation $y = -3x + 36$ yields $y = 6$, so the solution to the system of equations is $x = 10$, $y = 6$. The final step is left for the reader. It must be checked that $x = 10$ and $y = 6$ give true statements when substituted into the original system of equations.

To summarize the steps we followed to solve a system of linear equations in two variables using the algebraic method of substitution, we have:

Solving a System of Two Linear Equations in Two Variables using Substitution

1. Solve one equation for one variable.

2. Substitute the expression found in step 1 into the other equation.

3. Solve the resulting equation.

4. Substitute the value from step 3 back into the equation in step 1 to find the value of the remaining variable.

5. Check your solution!

6. Answer the question if it is a word problem.

A system of two linear equations in two variables may have one solution, no solutions, or infinitely many solutions. In Example 27.2 we will see a system with no solution.

Example 27.2. Solve the following system of equations by substitution.

$$\left(\begin{array}{l} x + y = 1 \\ y = -x + 2 \end{array} \right)$$

The second equation is already solved for y in terms of x so we can substitute it directly into $x + y = 1$:

$$x + (-x + 2) = 1 \Longrightarrow 2 = 1 \quad \text{False!}$$

Since we get the **false** statement $2 = 1$, the system of equations has **no** solution.

The Elimination Method

A second algebraic method for solving a system of linear equations is **the elimination method**. The basic idea of the method is to get the coefficients of one of the variables in the two equations to be additive inverses, such as -3 and 3, so that after the two equations are added, this variable is eliminated. Let's use one of the systems we solved in the previous section in order to illustrate the method:

$$\left(\begin{array}{ccccc} x & + & y & = & 7 \\ 5x & + & 10y & = & 40 \end{array} \right)$$

The coefficients of the x variable in our two equations are 1 and 5. We can make the coefficients of x to be additive inverses by multiplying the first equation by -5 and keeping the second equation untouched:

$$\left(\begin{array}{ccccc} x & + & y & = & 7 \\ 5x & + & 10y & = & 40 \end{array} \right) \Longrightarrow \left(\begin{array}{ccccc} (-5)(x & + & y) & = & (-5)7 \\ 5x & + & 10y & = & 40 \end{array} \right)$$

Using the distributive property, we rewrite the first equation as:

$$-5x \quad - \quad 5y \quad = \quad -35$$

Now we are ready to add the two equations to eliminate the variable x and solve the resulting equation for y:

$$
\begin{array}{rrrrr}
-5x & - & 5y & = & -35 \\
+ \quad 5x & + & 10y & = & 40 \\
\hline
 & & 5y & = & 5 \\
\Longrightarrow & & y & = & 1
\end{array}
$$

To find x, we can substitute $y = 1$ into either equation of the original system to solve for x:

$$x + 1 = 7 \quad \implies \quad x = 6$$

Hence, we get the same solution as we obtained using the substitution method in the previous section:

$$x = 6, \quad y = 1.$$

In this example, we only need to multiply the first equation by a number to make the coefficients of the variable x additive inverses. Sometimes, we need to multiply **both** equations by two different numbers to make the coefficients of one of the variables additive inverses. To illustrate this, let's look at Example 27.3.

Example 27.3.

$$\left(\begin{array}{ccccc} -3x & + & 2y & = & 3 \\ 4x & - & 3y & = & -6 \end{array} \right)$$

Lets aim to eliminate the y variable here. Since the least common multiple of 2 and 3 is 6, we can multiply the first equation by 3 and the second equation by 2, so that the coefficients of y are additive inverses:

$$\left(\begin{array}{ccccc} -3x & + & 2y & = & 3 \\ 4x & - & 3y & = & -6 \end{array} \right) \implies \left(\begin{array}{ccccc} (3)(-3x & + & 2y) & = & (3)3 \\ (2)(4x & - & 3y) & = & (2)(-6) \end{array} \right)$$

Using the distributive property, we rewrite the two equations as:

$$\left(\begin{array}{ccccc} -9x & + & 6y & = & 9 \\ 8x & - & 6y & = & -12 \end{array} \right)$$

Adding them together gives:

$$-1x = -3 \quad \implies \quad x = 3$$

To find y, we can substitute $x = 3$ into the first equation (or the second equation) of the original system to solve for y:

$$-3(3) + 2y = 3 \implies -9 + 2y = 3 \implies 2y = 12 \implies y = 6$$

Hence, the answer to the problem is:

$$x = 3 \quad y = 6$$

We can check the answer by substituting both numbers into the original system and see if both equations are correct.

The following steps summarize how to solve a system of equations by the elimination method:

Solving a System of Two Linear Equations in Two Variables using Elimination

1. Multiply one or both equations by a nonzero number so that the coefficients of one of the variables are additive inverses.

2. Add the equations to eliminate the variable.

3. Solve the resulting equation.

4. Substitute the value from step 3 back into either of the original equations to find the value of the remaining variable.

5. Check your solution!

6. Answer the question if it is a word problem.

Exit Problem

Solve algebraically:

$$
\begin{aligned}
8x &- 4y = 4 \\
3x &- 2y = 3
\end{aligned}
$$

Chapter 28

Solving a System of Linear Equations Graphically

We saw in chapter 27 how to solve a system of linear equations algebraically (by substitution and by elimination). Since we know that graphs of linear equations are *lines*, it is natural to graph the lines representing our system and observe where they are with respect to one another in the coordinate plane. There are only three possible configurations: the lines intersect at a single point, the lines coincide, the lines are parallel. Finding this intersection (if possible) amounts to solving the linear system.

Consider Example 27.1 from chapter 27.

Example 28.1.
$$\left(\begin{array}{ccccc} 6x & + & 2y & = & 72 \\ 3x & + & 8y & = & 78 \end{array} \right)$$

The solution we found (by substitution) was $x = 10$ and $y = 6$.

To solve this system by graphing, we start by graphing each of the given linear equations.

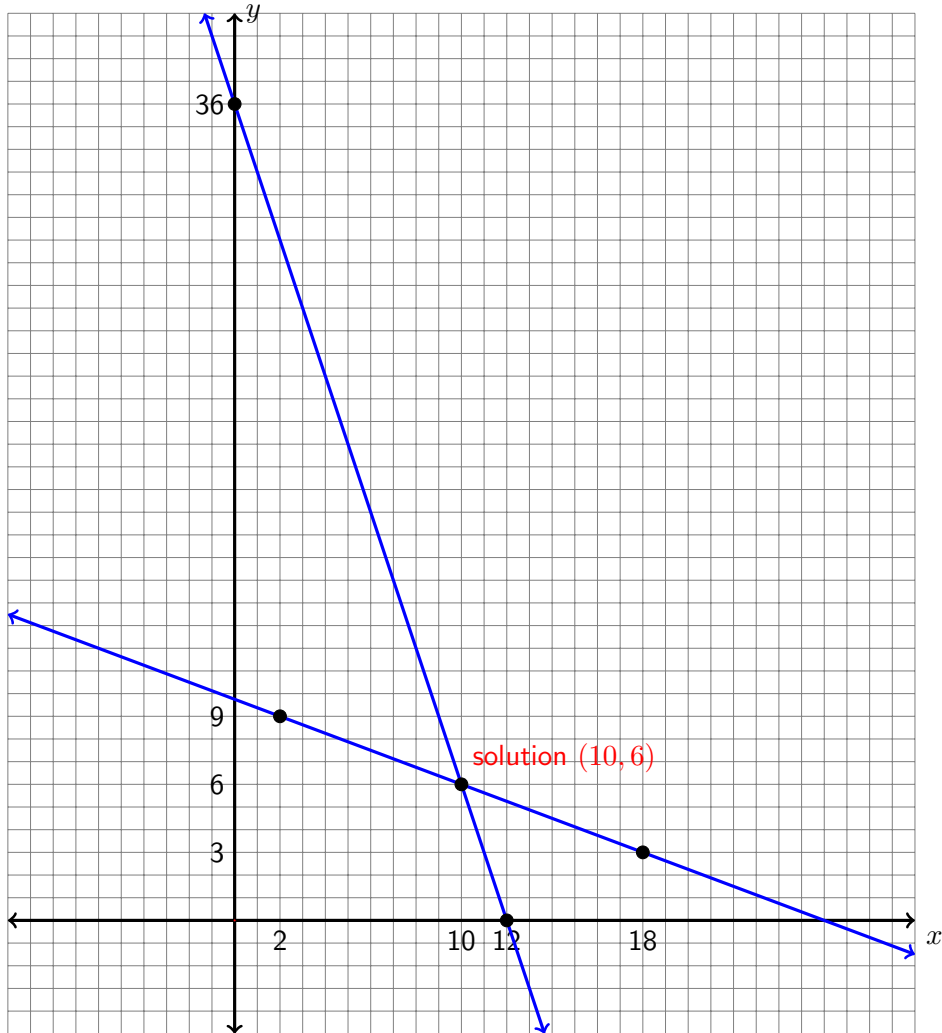First, we graph $6x + 2y = 72$. Remember, we find two points:

$$x = 0 \quad \implies \quad y = 36$$

$$y = 0 \quad \implies \quad x = 12$$

Similarly, we graph $3x + 8y = 78$ by finding two points on this line:

$$x = 2 \quad \implies \quad y = 9$$

$$y = 3 \quad \implies \quad x = 18$$

The lines are graphed in the same coordinate system as shown below.

The coordinates of the point of intersection of the two graphs are the solution of the system. Observe from the graph that the x and y coordinates of the point of intersection are $x = 10$ and $y = 6$.

Example 28.2. Solve by graphing:

$$\left(\begin{array}{ccccc} x & + & y & = & 7 \\ 5x & + & 10y & = & 40 \end{array} \right)$$

Step 1 Graph both lines:

$$x + y = 7: \qquad x = 0, y = 7$$

$$y = 0, x = 7$$

$$5x + 10y = 40: \qquad x = 0, y = 4$$

$$y = 0, x = 8$$

solution $(6, 1)$

Step 2 Read off the coordinates of the point of intersection:

$$x = 6, y = 1$$

Step 3 The solution of the system is $x = 6$, $y = 1$.

Remark 28.3.

1. If the graphs of the two lines do **not** have a point of intersection, then the system has no solution.

2. If the graphs of the lines **coincide**, then the system has infinitely many solutions.

Solving a System ot Two Linear Equations by Graphing

Step 1 Graph each linear equation on the same coordinate system.

Step 2 Label the point of intersection on the graph (if there is any).

Step 3 Read off the coordinates of the point of intersection (if possible). If the lines coincide, *every* point on them is considered an intersection point.

Step 4 Depending on Step 3, state whether the system has no solution (lines are parallel), infinitely many solutions (lines coincide), or one solution (single intersection point).

Exit Problem

Solve graphically:

$$
\begin{aligned}
8x &- 4y = 4 \\
3x &- 2y = 3
\end{aligned}
$$

Appendix A

The Multiplication Table

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Appendix B

Absolute Value

Every real number can be represented by a point on the real number line. The distance from a number (point) on the real line to the origin (zero) is what we called the **magnitude** (**weight**) of that number in Chapter 1. Mathematically, this is called the **absolute value** of the number. So, for example, the distance from the point -5 on the number line to the origin is 5 units.

Move 5 units to the *right*
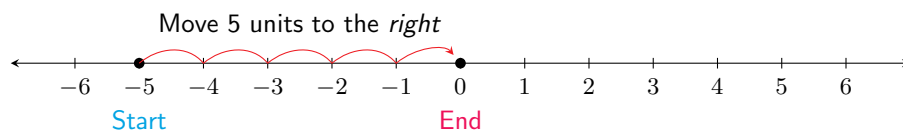
So is the distance from 5 to 0. Algebraically we can represent the absolute value as

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ x & \text{if } x > 0. \end{cases}$$

Example B.1. Evaluate each expression:

a) $|7| = 7$

b) $|-7| = 7$

c) $\left|\dfrac{-1}{4}\right| = \dfrac{1}{4}$

d) $|-3| + |2| = 3 + 2 = 5$

e) $|-3 + 2| = |-1| = 1$

f) $|3| - |2| = 3 - 2 = 1$

g) $\dfrac{|-16|}{|-4|} = \dfrac{16}{4} = 4$

h) $\dfrac{-|7| - |-5|}{-|-3|} = \dfrac{-7 - 5}{-3} = \dfrac{-12}{-3} = 4$

i) $10 \cdot \dfrac{|3^2 - 3|}{4} + 2 = 10 \cdot \dfrac{|9 - 3|}{4} + 2 = 10 \cdot \dfrac{|6|}{4} + 2 = 10 \cdot \dfrac{6}{4} + 2$

$= \cancel{2} \cdot 5 \cdot \dfrac{6}{\cancel{2} \cdot 2} + 2 = 5 \cdot \dfrac{6}{2} + 2 = 5 \cdot 3 + 2 = 15 + 2 = 17$

Note In relation to the order of operations PE(MD)(AS), the absolute value symbol is treated as a *parenthesis*, and so, what is inside has the priority over other operations.

Appendix C

Formulas from Geometry

We review some well-known formulas from planar geometry.

The rectangle with length l and width w:
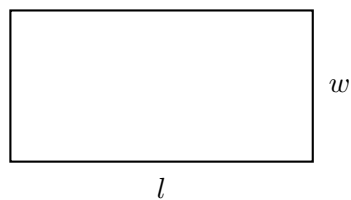
$$\boxed{\text{Area: } A = lw}$$

$$\boxed{\text{Perimeter: } P = 2l + 2w}$$

(diagram: rectangle with side w and base l)

The square with side length a:

$$\boxed{\text{Area: } A = a^2}$$

$$\boxed{\text{Perimeter: } P = 4a}$$
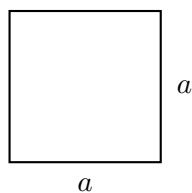
(diagram: square with sides a and a)

The triangle with base b and height h, and with side lengths a, b, c:
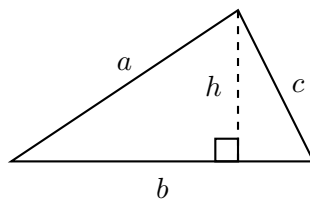
$$\boxed{\text{Area: } A = \tfrac{1}{2}bh}$$

$$\boxed{\text{Perimeter: } P = a + b + c}$$

(diagram: triangle with sides a, c, height h, and base b)

The circle with radius r and diameter d:
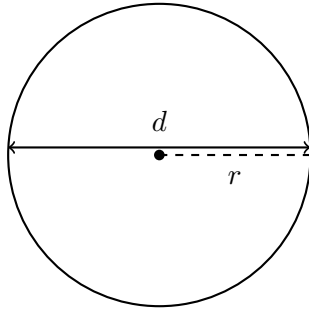
Diameter: $d = 2r$
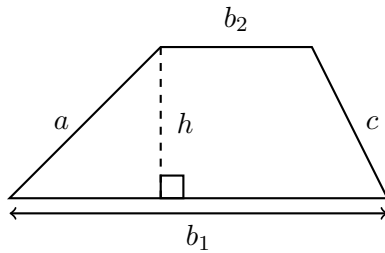
Area: $A = \pi r^2$

Circumference: $C = 2\pi r$

The parallelogram with base b and height h, and side lengths a and b:
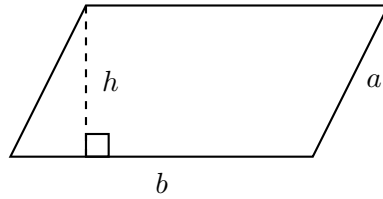
Area: $A = bh$

Perimeter: $P = 2a + 2b$

The trapezoid with bases b_1 and b_2, height h, and side lengths a, c, b_1, b_2:
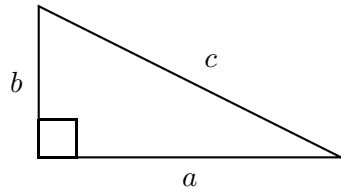

Area: $A = \frac{1}{2}(b_1 + b_2)h$

Perimeter: $P = a + b_1 + c + b_2$

Appendix D

The Pythagorean Theorem

The Pythagorean Theorem relates the lengths of the legs of a right triangle and the hypotenuse.

The Pythagorean Theorem: If a and b are the lengths of the legs of the right triangle and c is the length of the hypotenuse (the side opposite the right angle) as seen in this figure,


then

$$a^2 + b^2 = c^2.$$

Proof. You can verify the truth of the Pythagorean Theorem by measuring various triangles and checking to see that they satisfy $a^2 + b^2 = c^2$ where c is the length of the hypotenuse and a and b are the lengths of the legs. However, you would have to measure every such triangle to know its absolute truth which, of course, is impossible. We therefore would like to find a way to see it is true for all right triangles. The Pythagorean Theorem can be seen to be true by considering the following construction. Start with the triangle (any right triangle can be oriented this way)
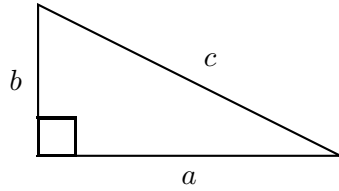
then construct three other triangles which are the same size and shape as the first (i.e., congruent to the first):
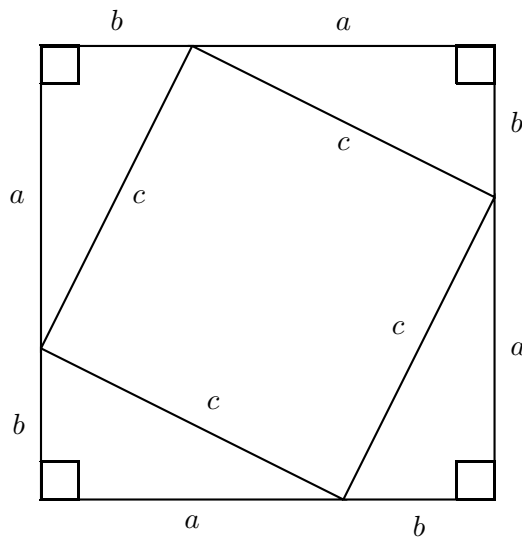
Now, compute the area of the large square in two ways:

1. Using the area of a square formula:

$$\text{Area} = (a + b)^2 = a^2 + 2ab + b^2$$

2. Adding up the sum of the four triangles plus the little square inside:

$$\text{Area} = 4 \cdot \left(\frac{1}{2}ab\right) + c^2 = 2ab + c^2.$$

Since both are equal to the area of the square, we must have

$$a^2 + 2ab + b^2 = 2ab + c^2.$$

Subtracting $2ab$ from both sides gives

$$a^2 + b^2 = c^2.$$

$\square$