SACRAMENTO STATE
Redefine the Possible

A Reduced-Order Model for the Dynamics of a Flexible Solar Sail

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INTRODUCTION

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- designing a feedback control to stabilize the solarcraft about the desired maneuver

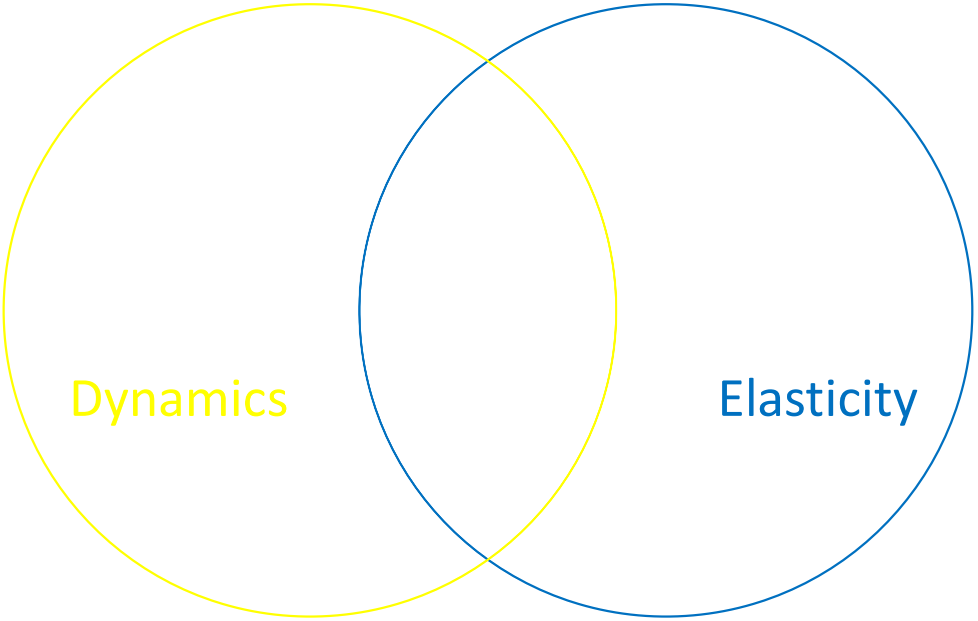
INTRODUCTION

Flexible solarcraft motion involves 3 fundamental disciplines:

INTRODUCTION

Dynamics

INTRODUCTION



Dynamics

Elasticity

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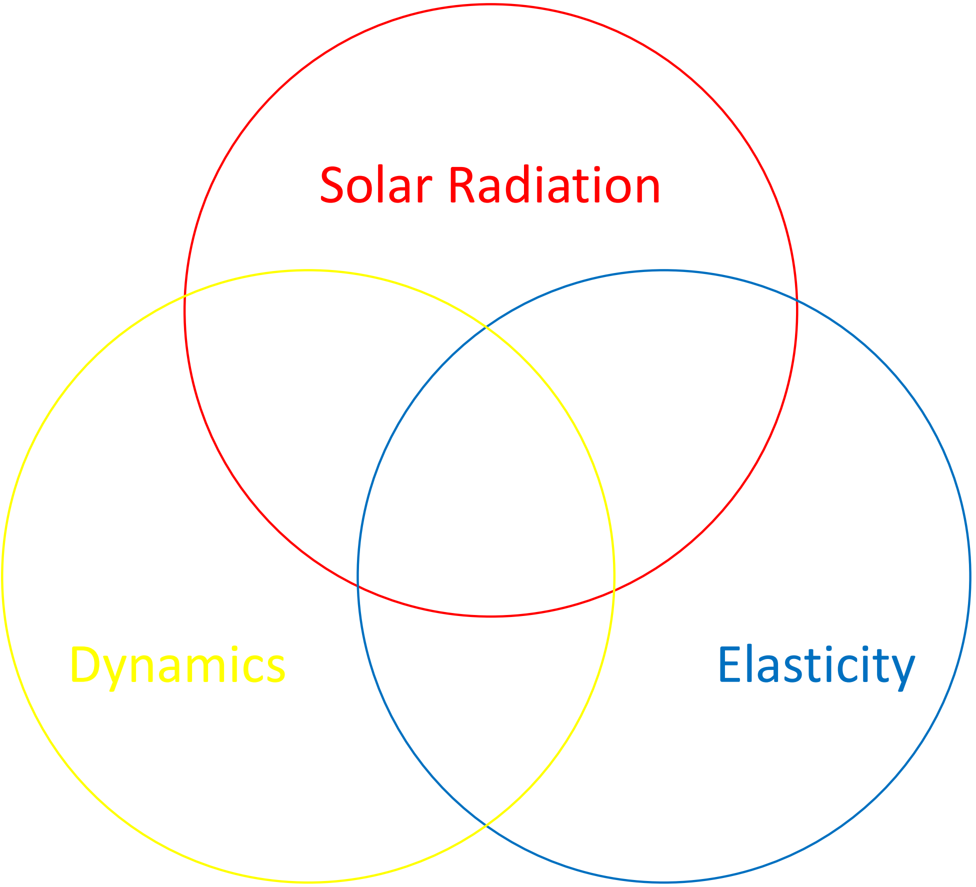
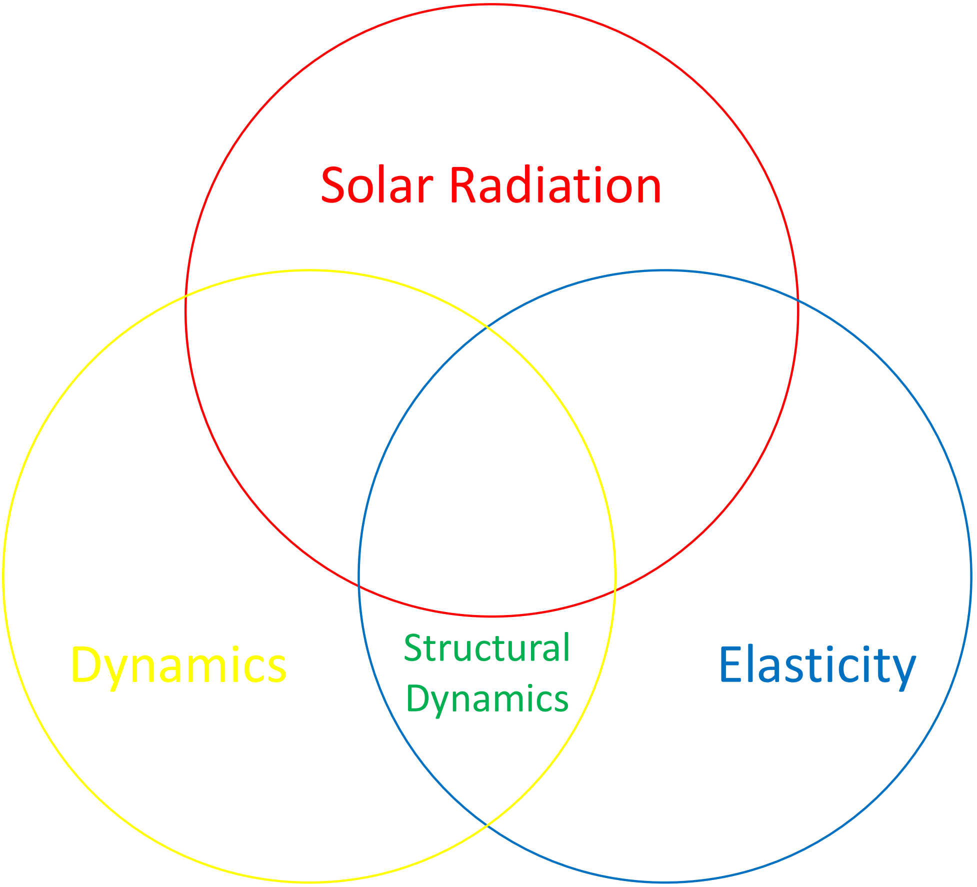


Solar Radiation

Dynamics

Elasticity

INTRODUCTION



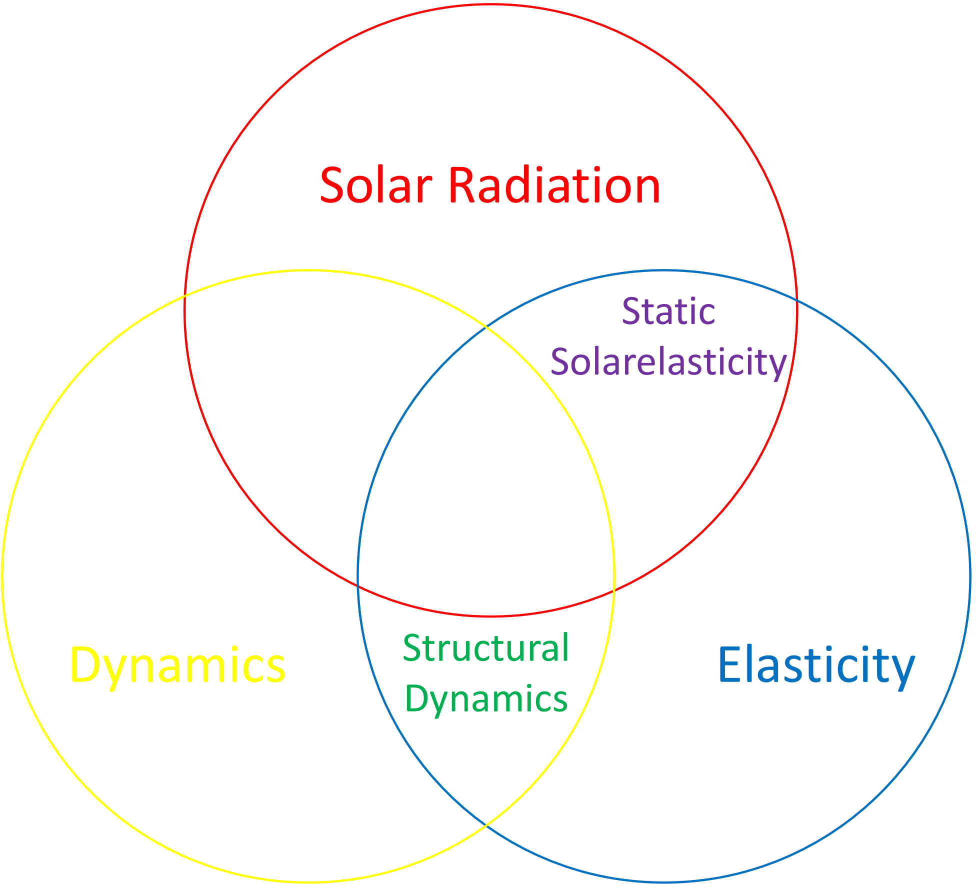
Solar Radiation

Dynamics

Structural Dynamics

Elasticity

INTRODUCTION



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Static Solarelasticity

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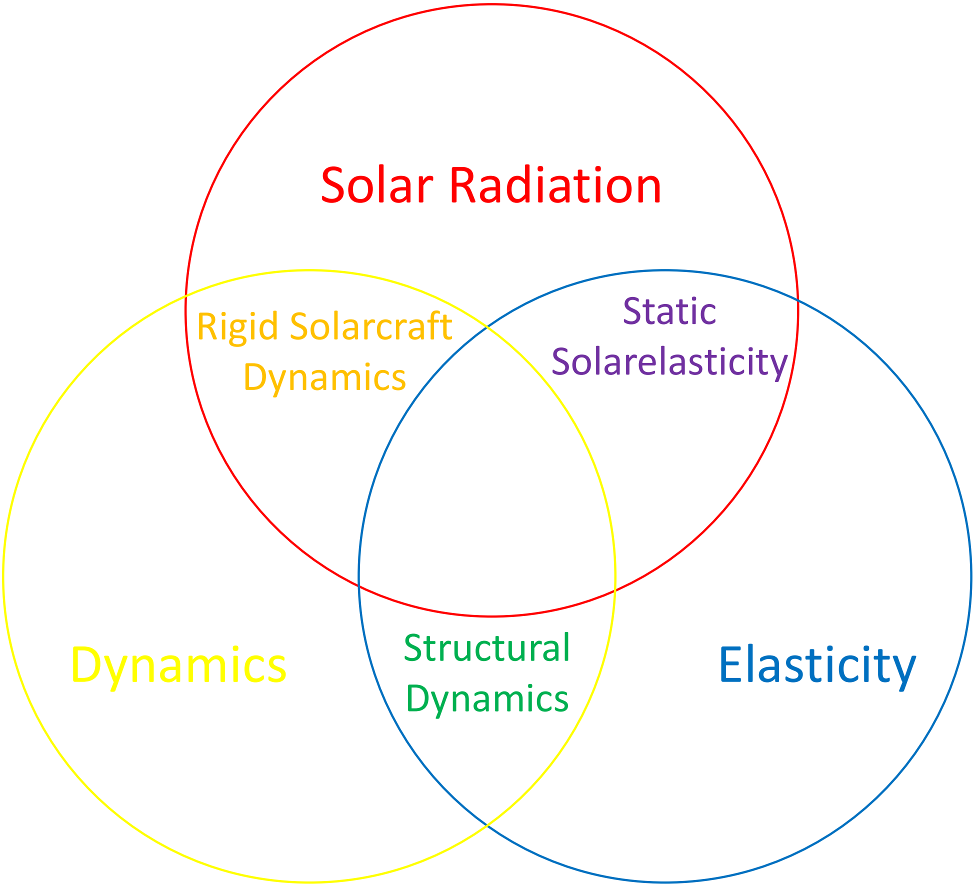

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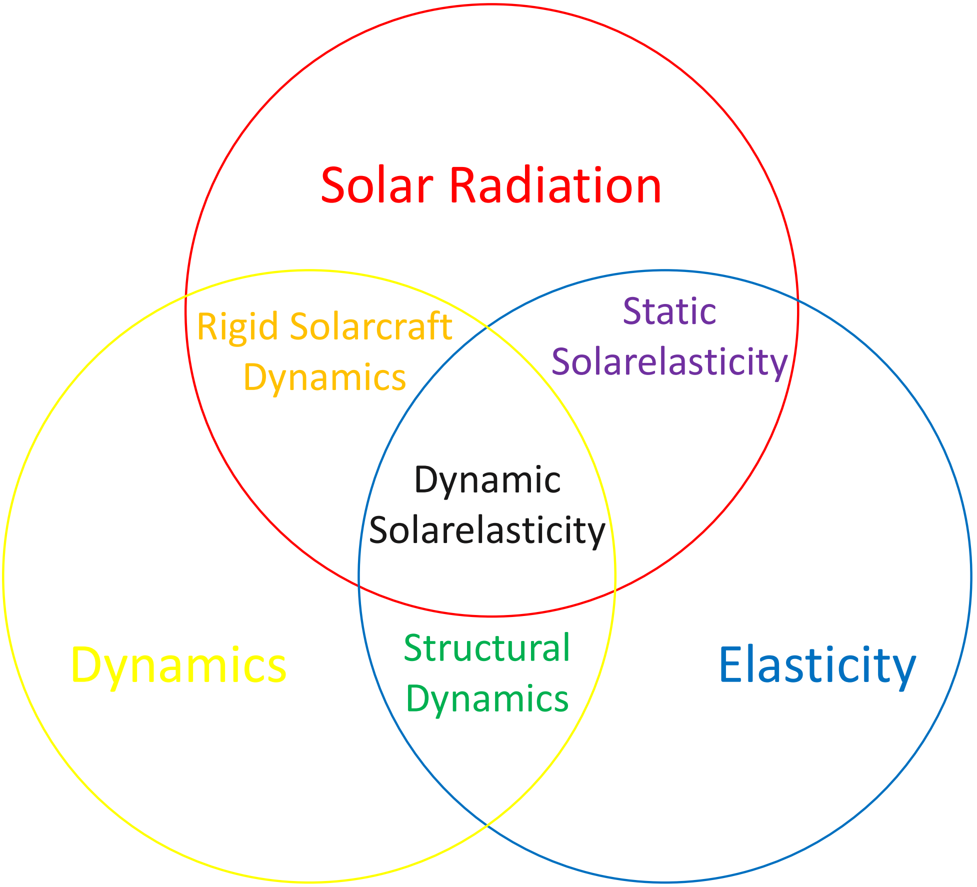

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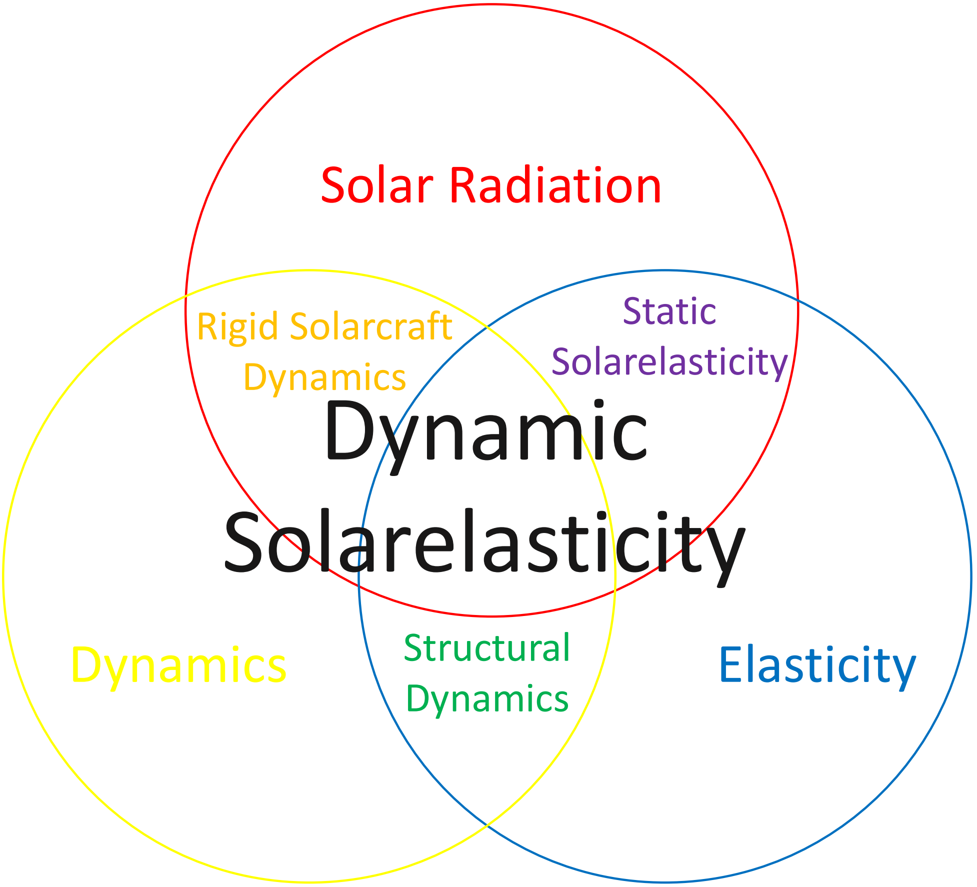

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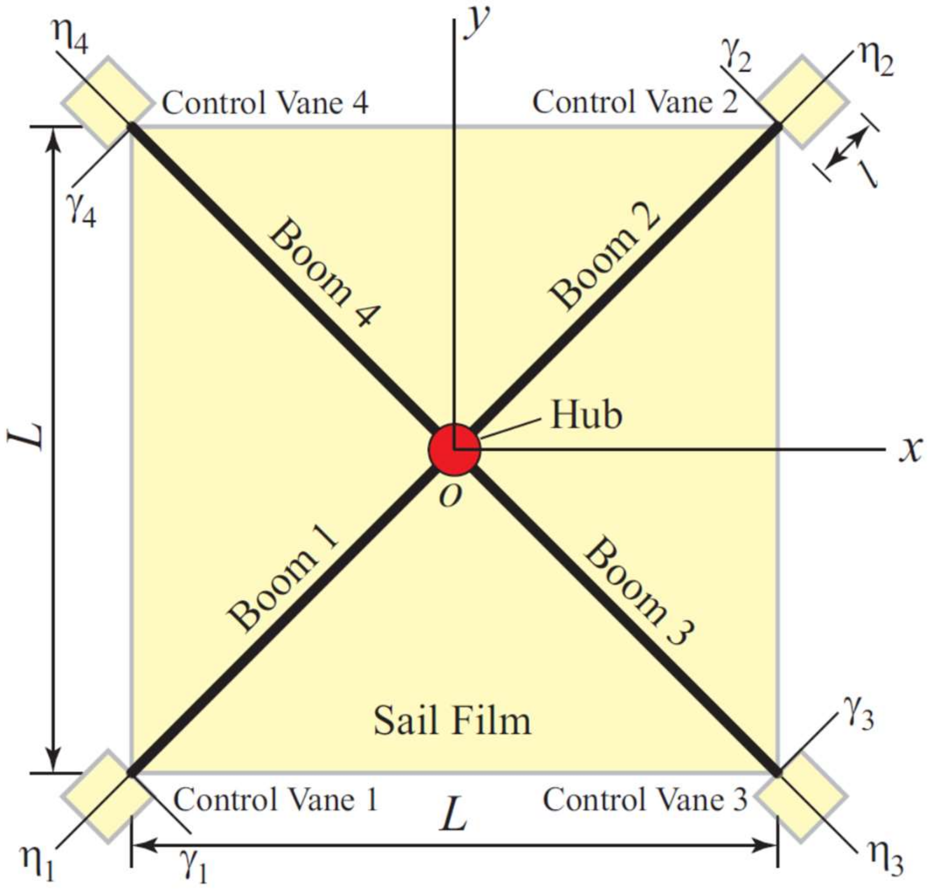

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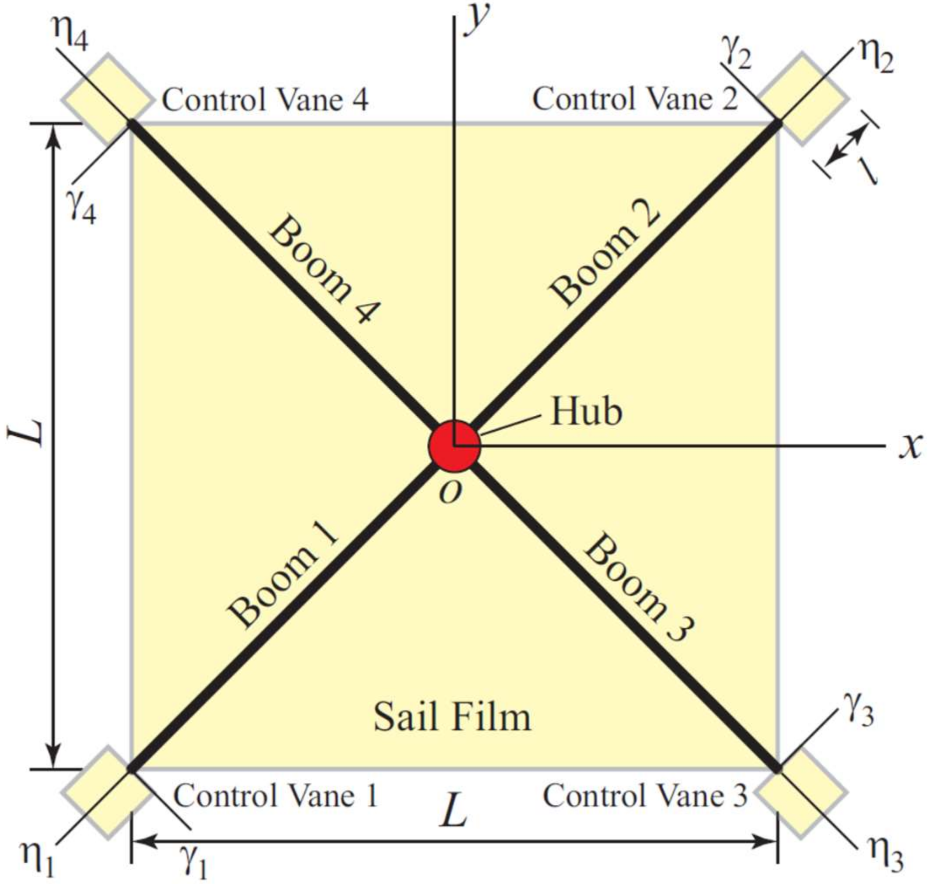
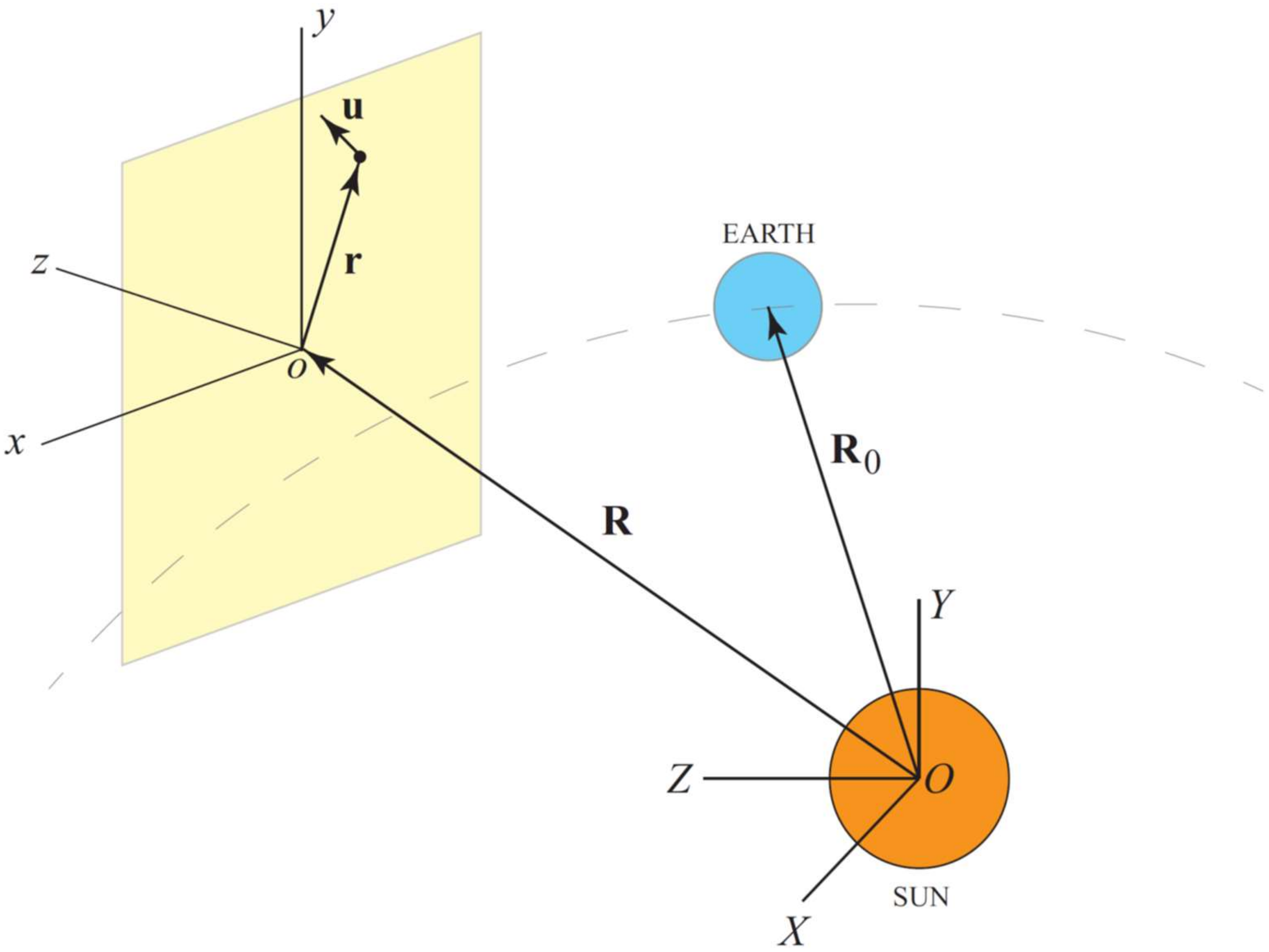


We assume the sail film is connected to the booms continuously.

EQUATIONS OF MOTION



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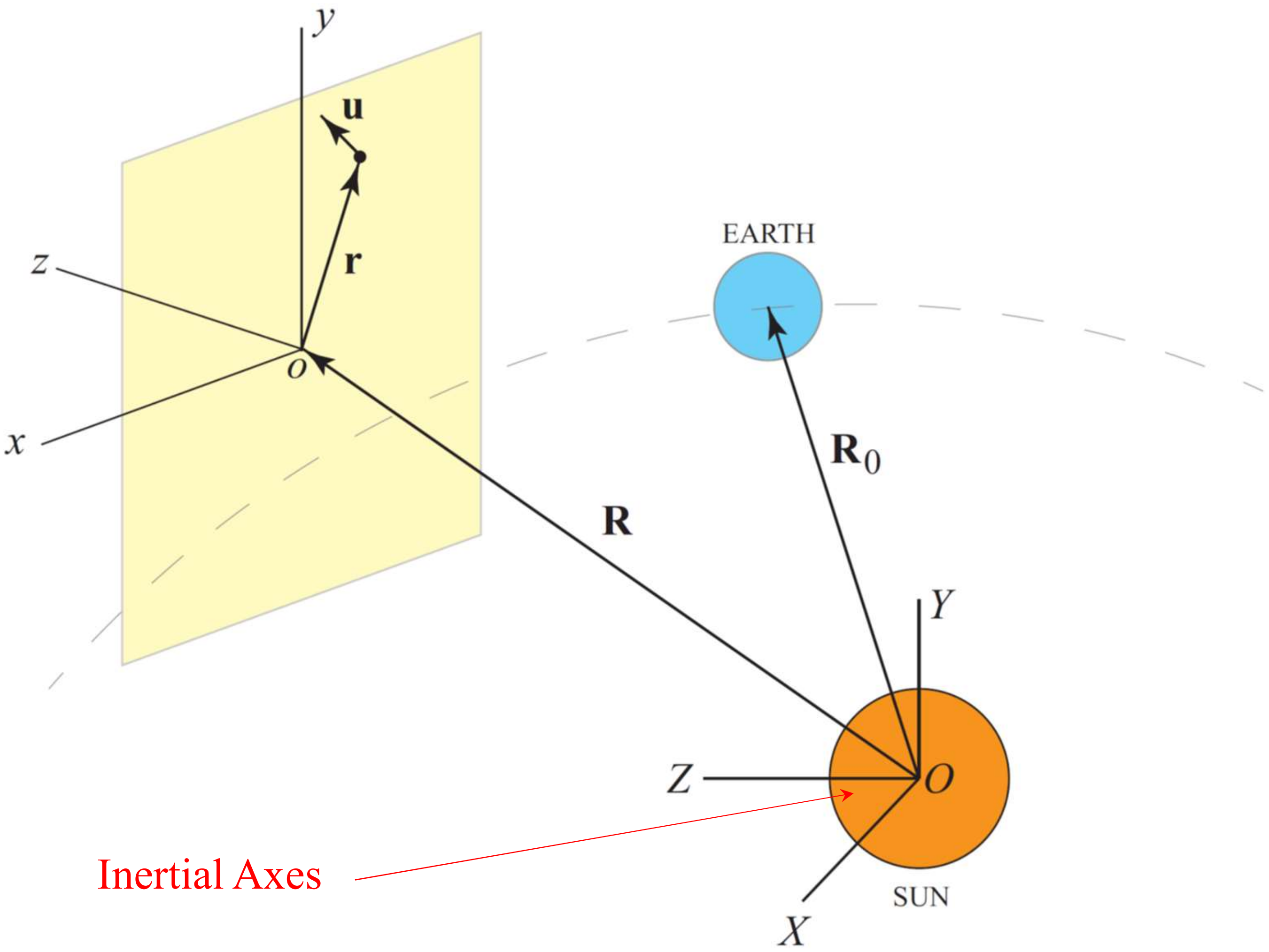
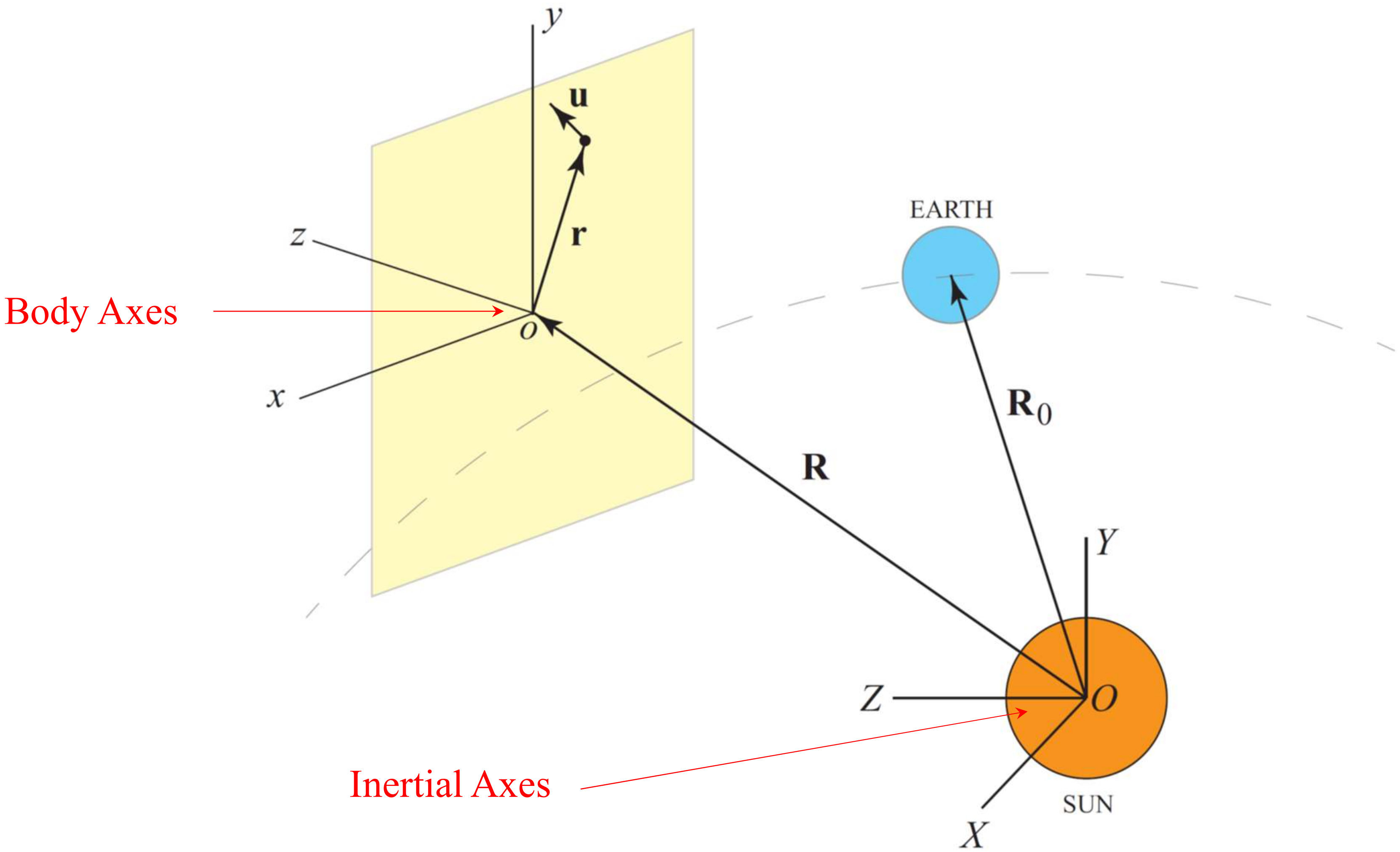


Inertial Axes

EQUATIONS OF MOTION



Body Axes

Inertial Axes

SACRAMENTO STATE
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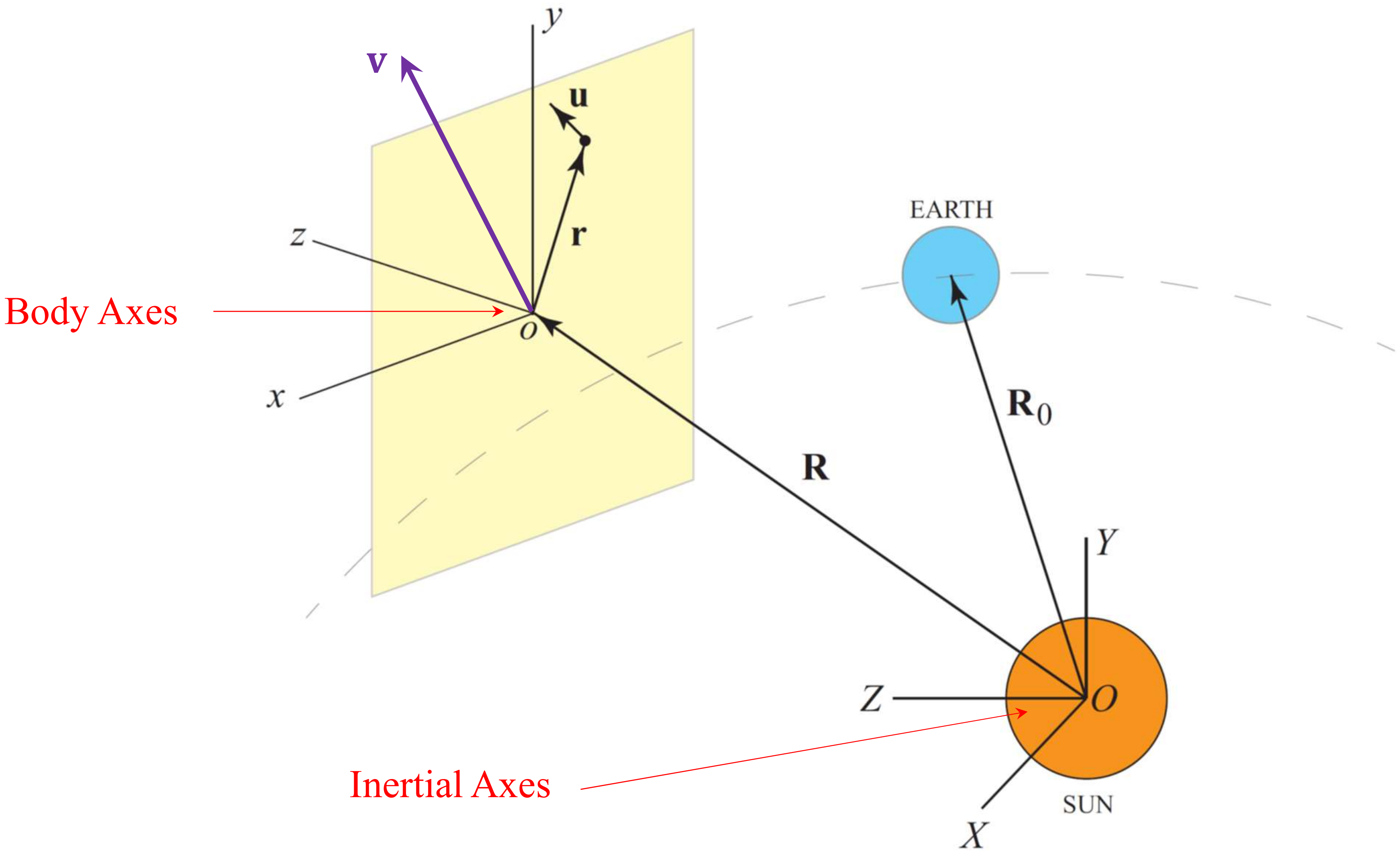
EQUATIONS OF MOTION



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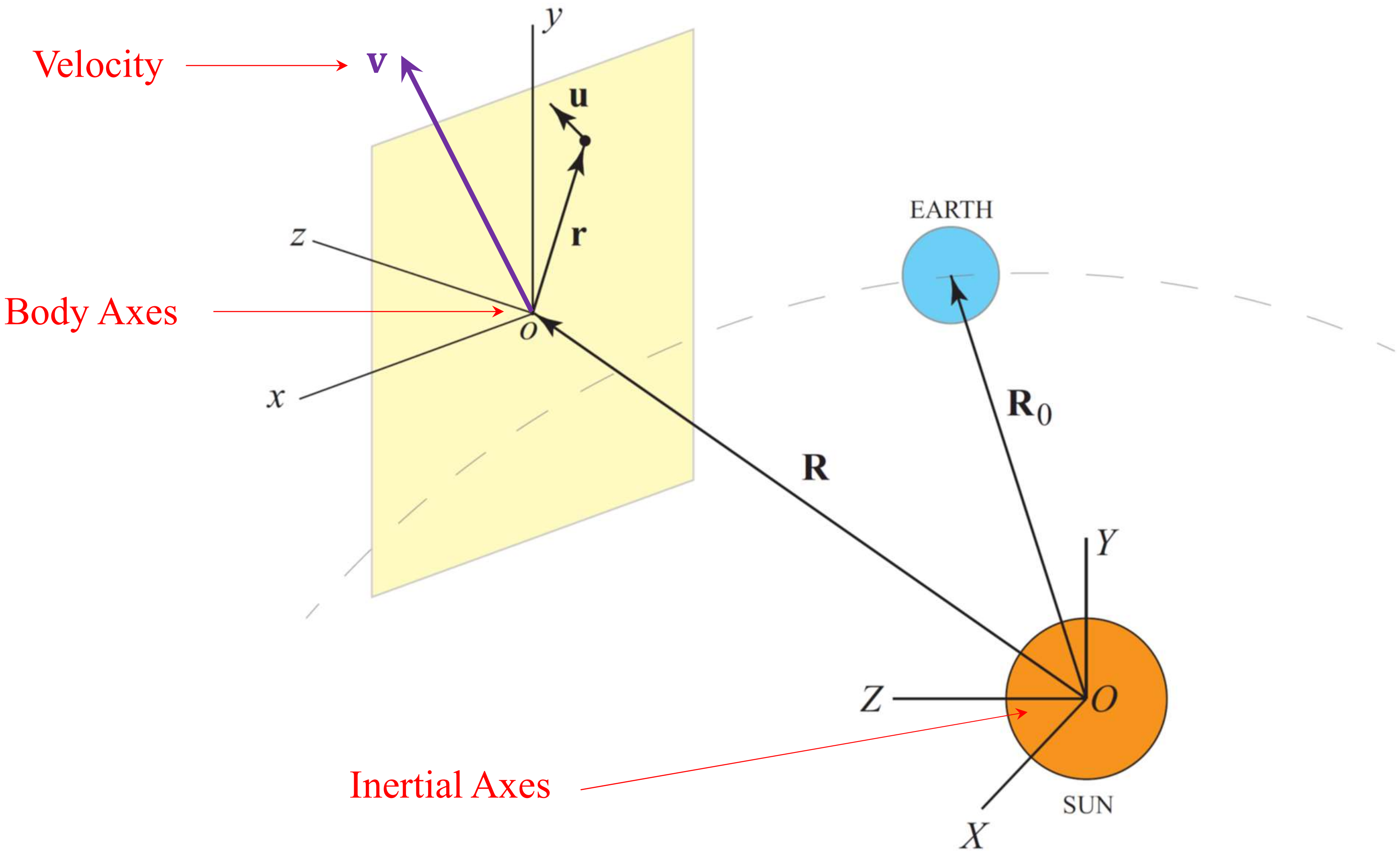
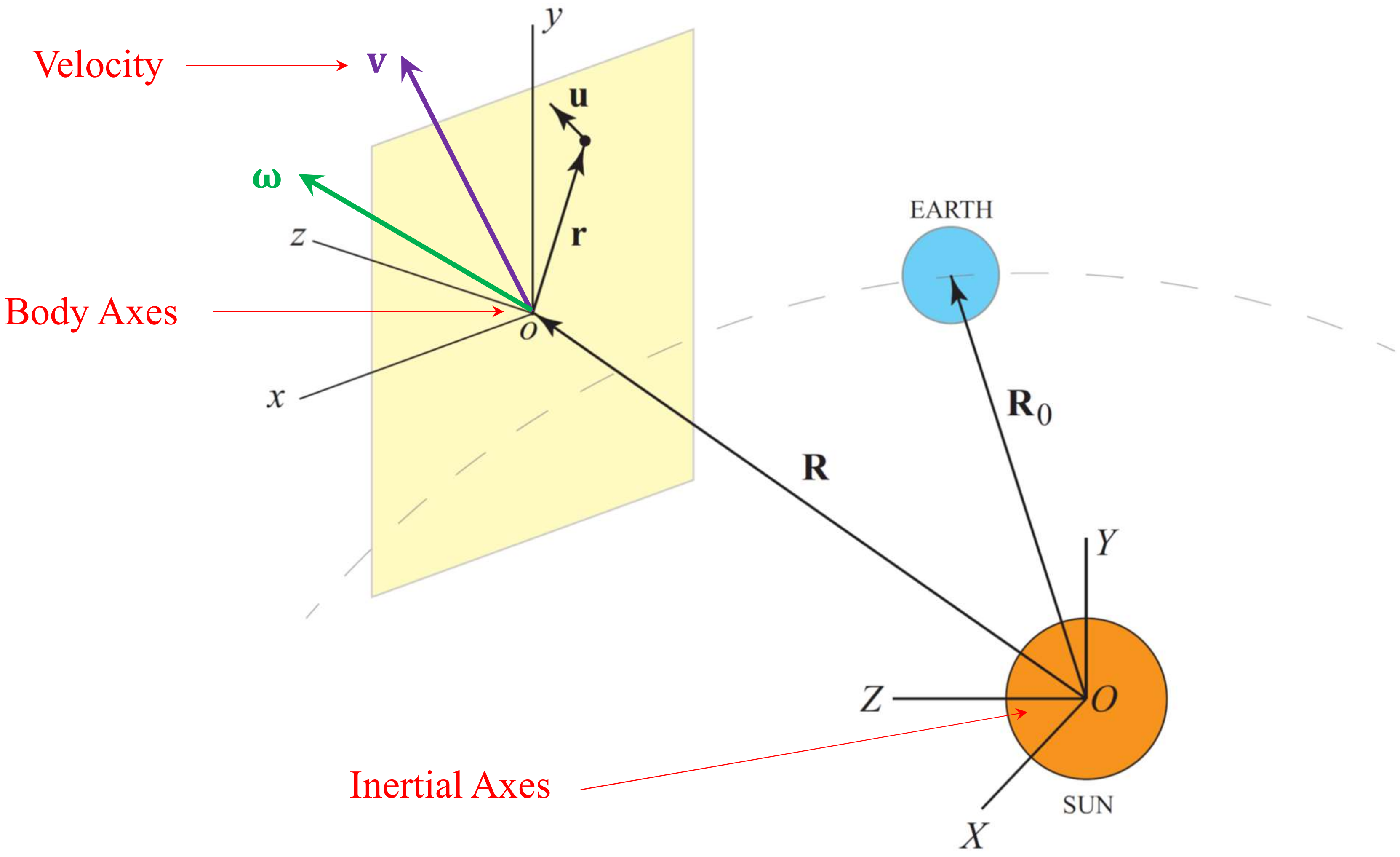


Velocity

Body Axes

Inertial Axes

EQUATIONS OF MOTION



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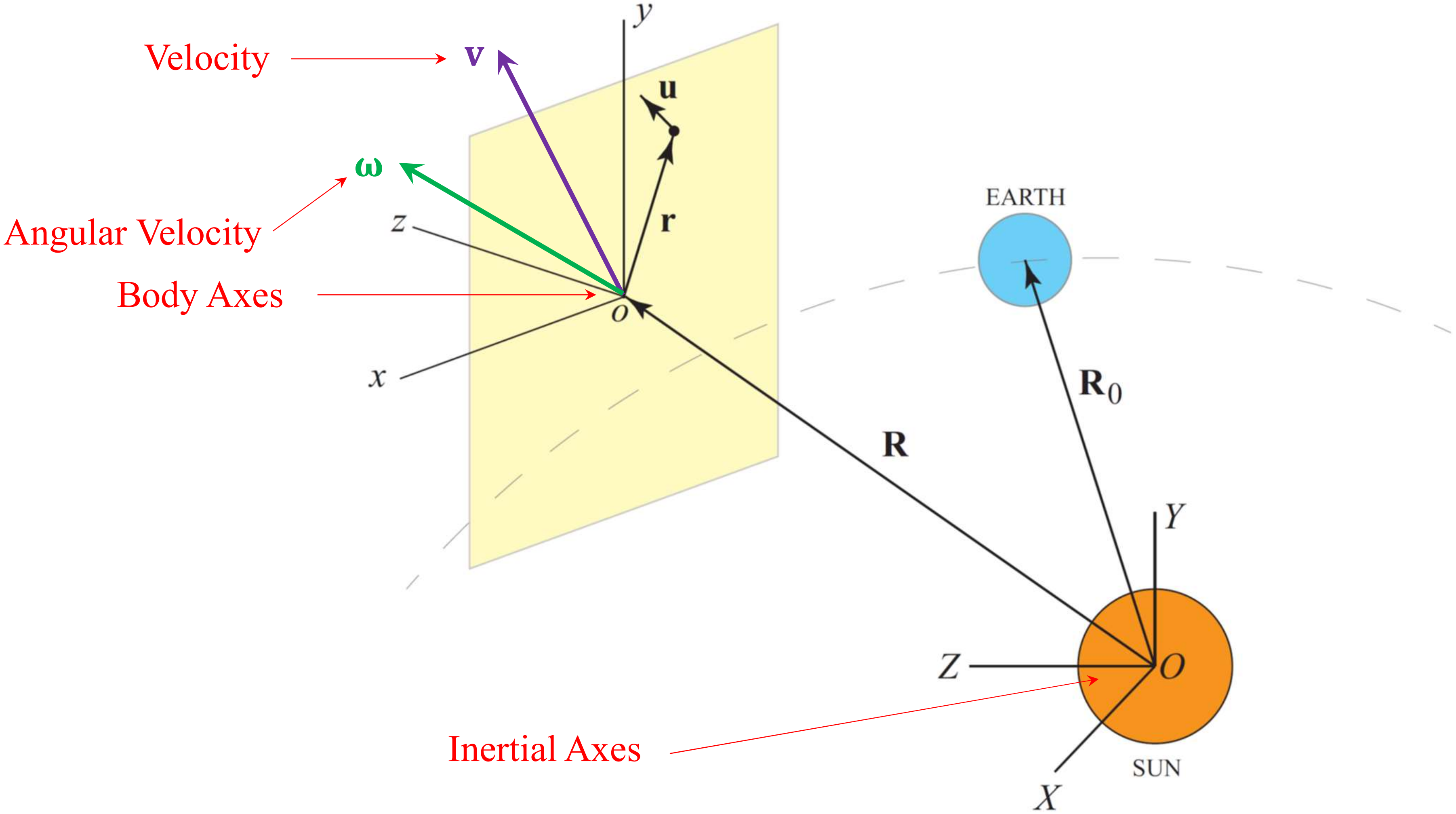

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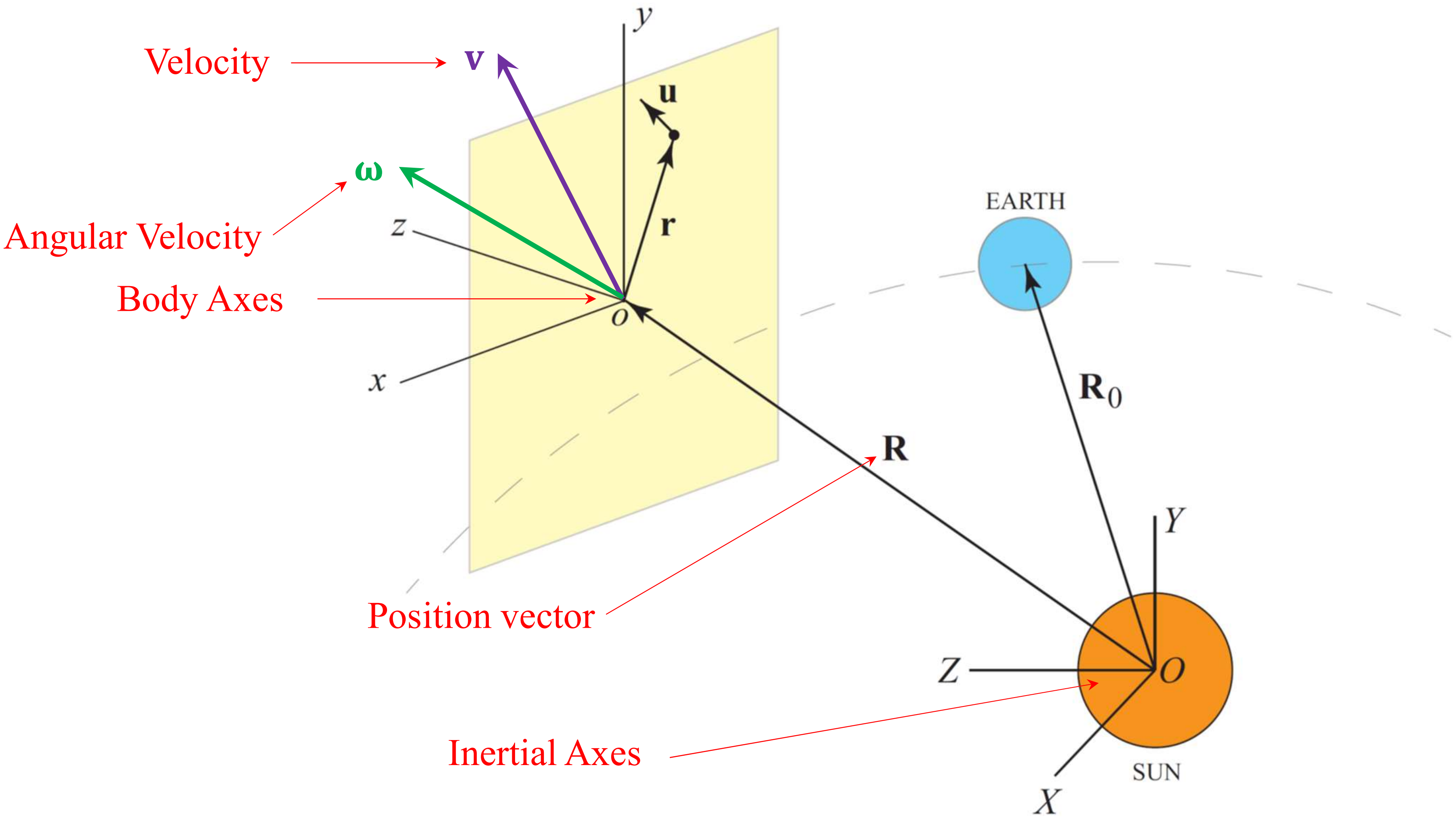


Velocity

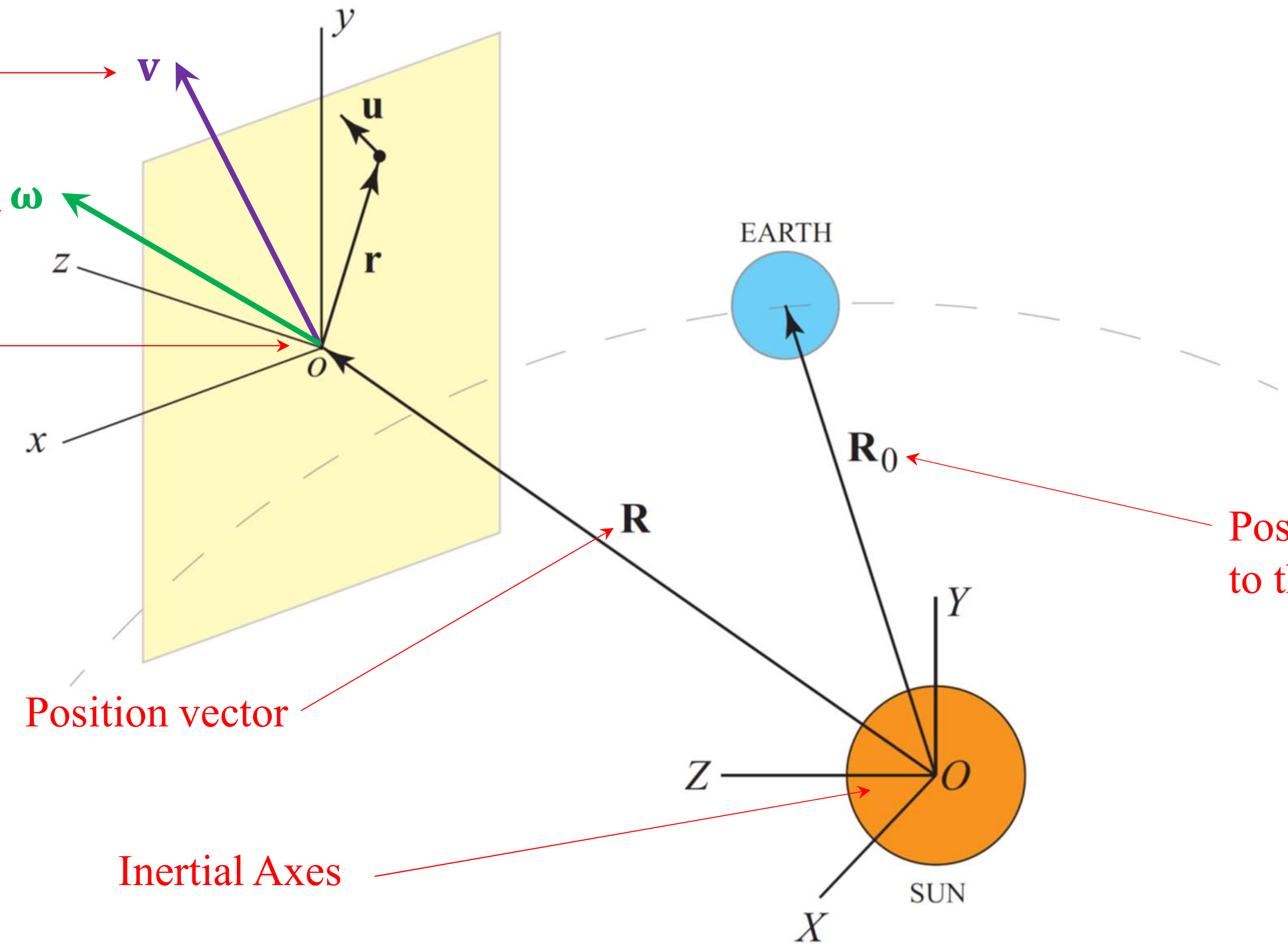
Angular Velocity

Body Axes

Position vector

Inertial Axes

EQUATIONS OF MOTION



Velocity

Angular Velocity

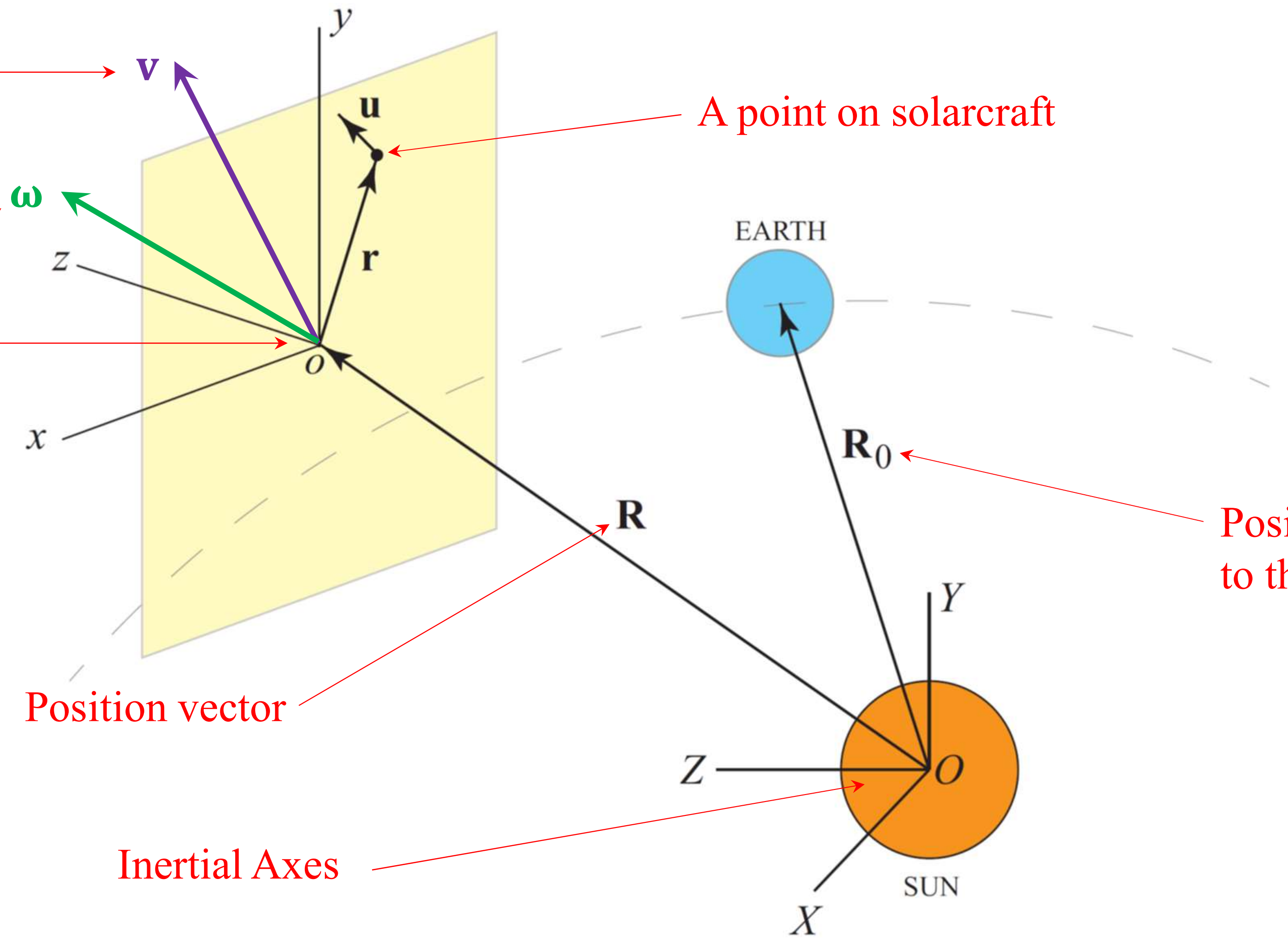
Body Axes

Position vector

Inertial Axes

Position vector
to the Earth

EQUATIONS OF MOTION



Velocity

A point on solarcraft

Angular Velocity

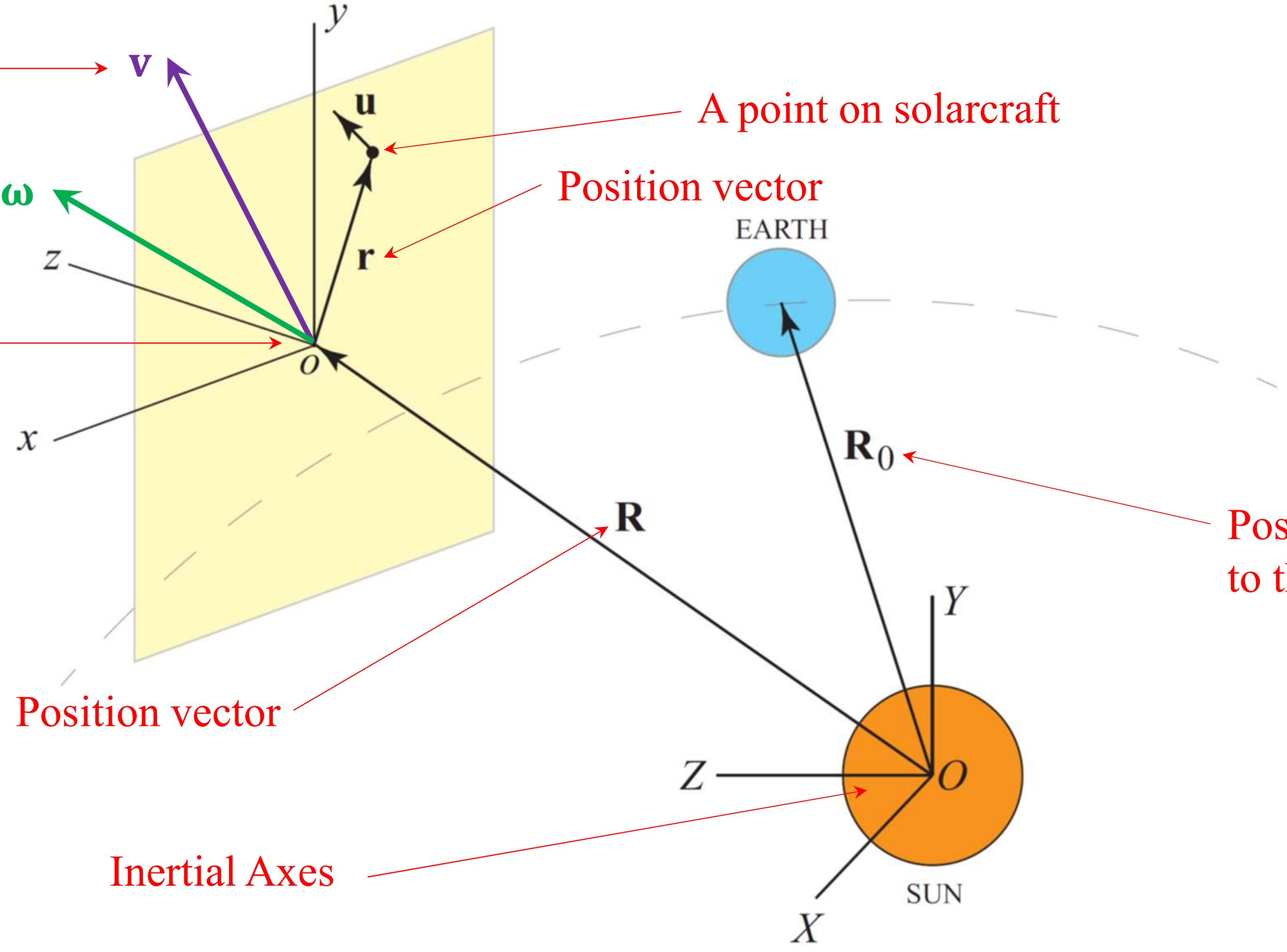
Body Axes

Position vector

Position vector
to the Earth

Inertial Axes

EQUATIONS OF MOTION



Velocity

A point on solarcraft

Position vector

Angular Velocity

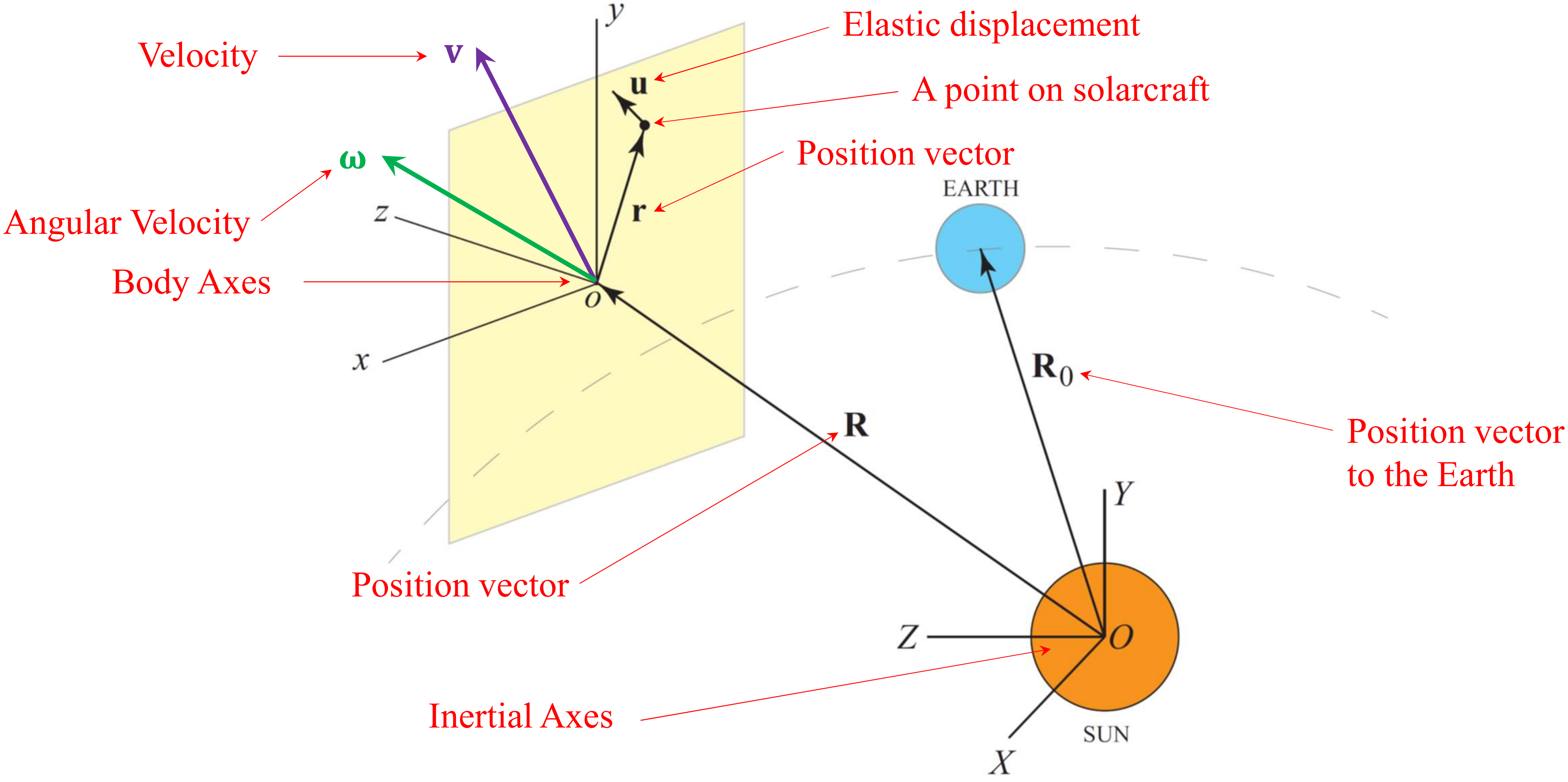
Body Axes

Position vector
to the Earth

Position vector

Inertial Axes

EQUATIONS OF MOTION



Velocity

Elastic displacement

A point on solarcraft

Position vector

Angular Velocity

Body Axes

Position vector to the Earth

Position vector

Inertial Axes

EQUATIONS OF MOTION

Body axes xyz are obtained from the inertial axes XYZ:

1. Rotation θ about Y to the axes $x_1 y_1 z_1$
2. Rotation ϕ about x_1 to the axes $x_2 y_2 z_2$
3. Rotation ψ about z_2 to the body axes xyz

ϕ, θ and ψ are the Euler angles.

EQUATIONS OF MOTION

The velocity of the origin o of the body axes $\mathbf{v} = [v_x \quad v_y \quad v_z]^T$.

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where $\boldsymbol{\theta} = [\phi \quad \theta \quad \psi]^T$.

EQUATIONS OF MOTION

v and **ω** are given in body axes components.

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$\mathbf{v}$ and $\boldsymbol{\omega}$ are referred to as *quasi-velocities*.

EQUATIONS OF MOTION

Local axes for Control Vane 1-4 are obtained from xyz:

1. Rotation α_i about z to the axes $x_i' y_i' z_i'$
2. Rotation β_i about x_i' to the axes $x_i'' y_i'' z_i''$
3. Rotation δ_i about y_i'' to the local axes $\eta_i \gamma_i \kappa_i$

for $i = 1,2,3,4.$

For the square solar sail, $\alpha_1 = 225°, \alpha_2 = 45°, \alpha_3 = 315°, \alpha_4 = 135°.$

EQUATIONS OF MOTION

Equations of motion of flexible solar sail can be obtained by means of the Lagrange’s equations of motion in quasi-coordinates.

SACRAMENTO STATE
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$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}}\right) + \tilde{\omega}\frac{\partial \mathcal{L}}{\partial \mathbf{v}} - C\frac{\partial \mathcal{L}}{\partial \mathbf{R}} &= \mathbf{F} \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}}\right) + \tilde{v}\frac{\partial \mathcal{L}}{\partial \mathbf{v}} + \tilde{\omega}\frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}} - (E^T)^{-1}\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} &= \mathbf{M} \\
\frac{\partial}{\partial t}\left(\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\mathbf{u}}}\right) - \frac{\partial \hat{\mathcal{L}}}{\partial \mathbf{u}} + \frac{\partial \hat{\mathcal{R}}}{\partial \dot{\mathbf{u}}} &= \hat{\mathbf{U}}
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\end{aligned}$$

where $\mathcal{L} = \mathcal{T} - \mathcal{V}$ is the Lagrangian

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Equations of motion of flexible solar sail can be obtained by means of the Lagrange’s equations of motion in quasi-coordinates.

$$
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Kinetic energy

Potential energy

EQUATIONS OF MOTION

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Generalized Forces

where $\mathcal{L} = \mathcal{T} - \mathcal{V}$ is the Lagrangian

Kinetic energy

Potential energy

EQUATIONS OF MOTION

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$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\omega}}\right)+\tilde{v}\frac{\partial\mathcal{L}}{\partial\mathbf{v}}+\tilde{\omega}\frac{\partial\mathcal{L}}{\partial\boldsymbol{\omega}}-(E^{T})^{-1}\frac{\partial\mathcal{L}}{\partial\boldsymbol{\theta}}=\mathbf{M}$$

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Kinetic energy

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EQUATIONS OF MOTION

Equations of motion of flexible solar sail can be obtained by means of the Lagrange's equations of motion in quasi-coordinates.

Force Equations:
3 nonlinear ODEs

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\begin{aligned}
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\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}}\right) + \tilde{v}\frac{\partial \mathcal{L}}{\partial \mathbf{v}} + \tilde{\omega}\frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}} - (E^T)^{-1}\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} &= \mathbf{M} \\
\frac{\partial}{\partial t}\left(\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\mathbf{u}}}\right) - \frac{\partial \hat{\mathcal{L}}}{\partial \mathbf{u}} + \frac{\partial \hat{\mathcal{R}}}{\partial \dot{\mathbf{u}}} &= \hat{\mathbf{U}}
\end{aligned}
$$

Generalized
Forces

where $\mathcal{L} = \mathcal{T} - \mathcal{V}$ is the Lagrangian

Kinetic energy

Potential energy

EQUATIONS OF MOTION

Equations of motion of flexible solar sail can be obtained by means of the Lagrange's equations of motion in quasi-coordinates.

Force Equations:
3 nonlinear ODEs

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Equations governing
elastic displacement:
3 nonlinear PDEs

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\mathbf{u}}}\right) - \frac{\partial \hat{\mathcal{L}}}{\partial \mathbf{u}} + \frac{\partial \hat{\mathcal{R}}}{\partial \dot{\mathbf{u}}} = \hat{\mathbf{U}}$$

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The elastic displacement

$$\mathbf{u} = [0 \quad 0 \quad w]^T$$

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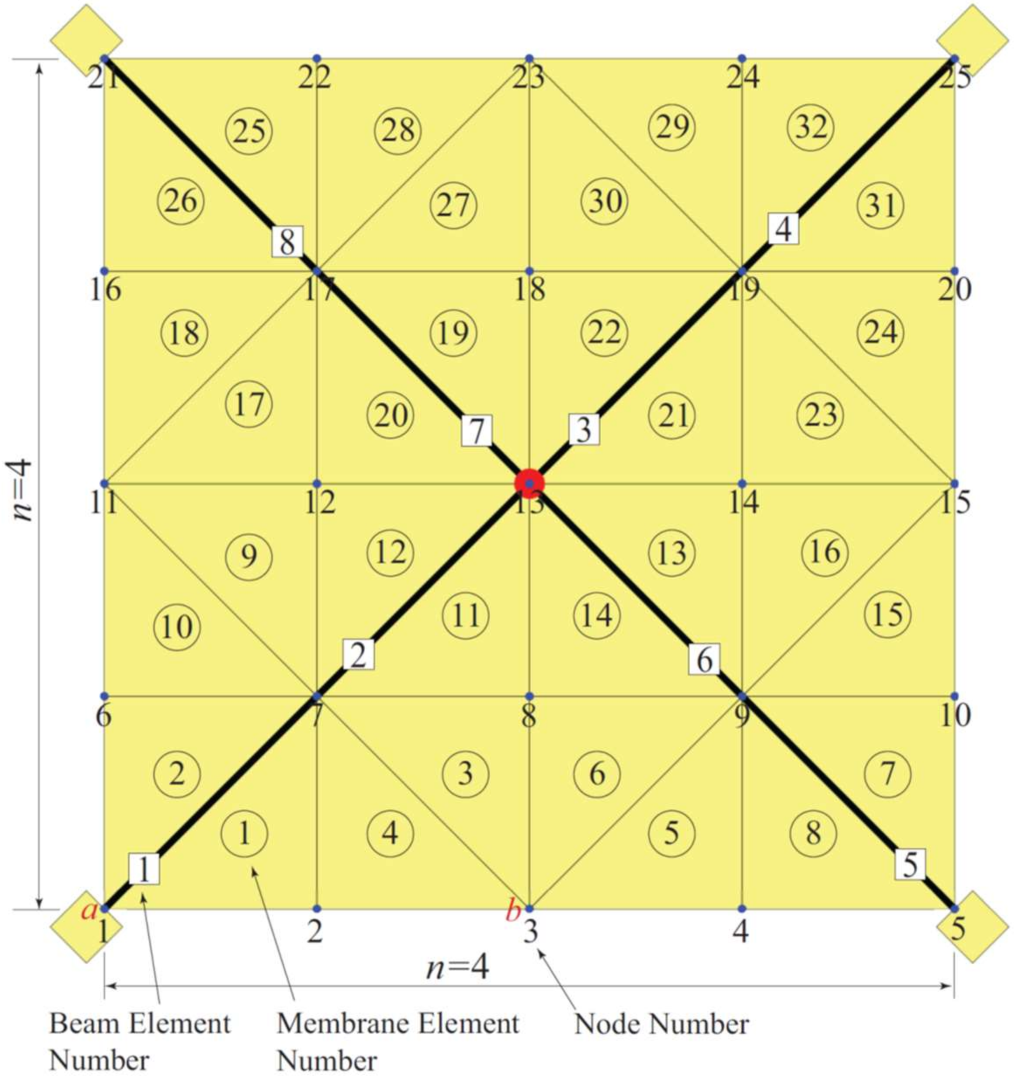

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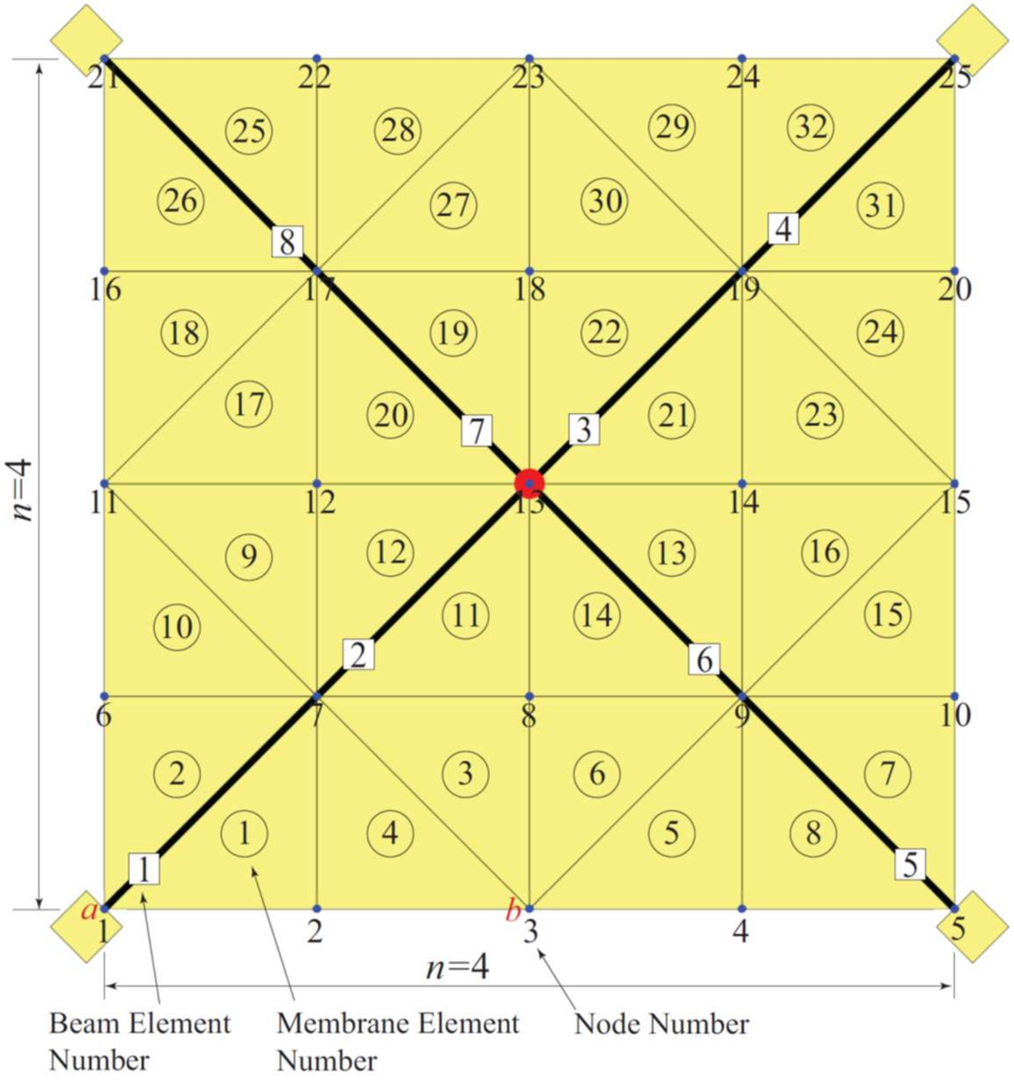

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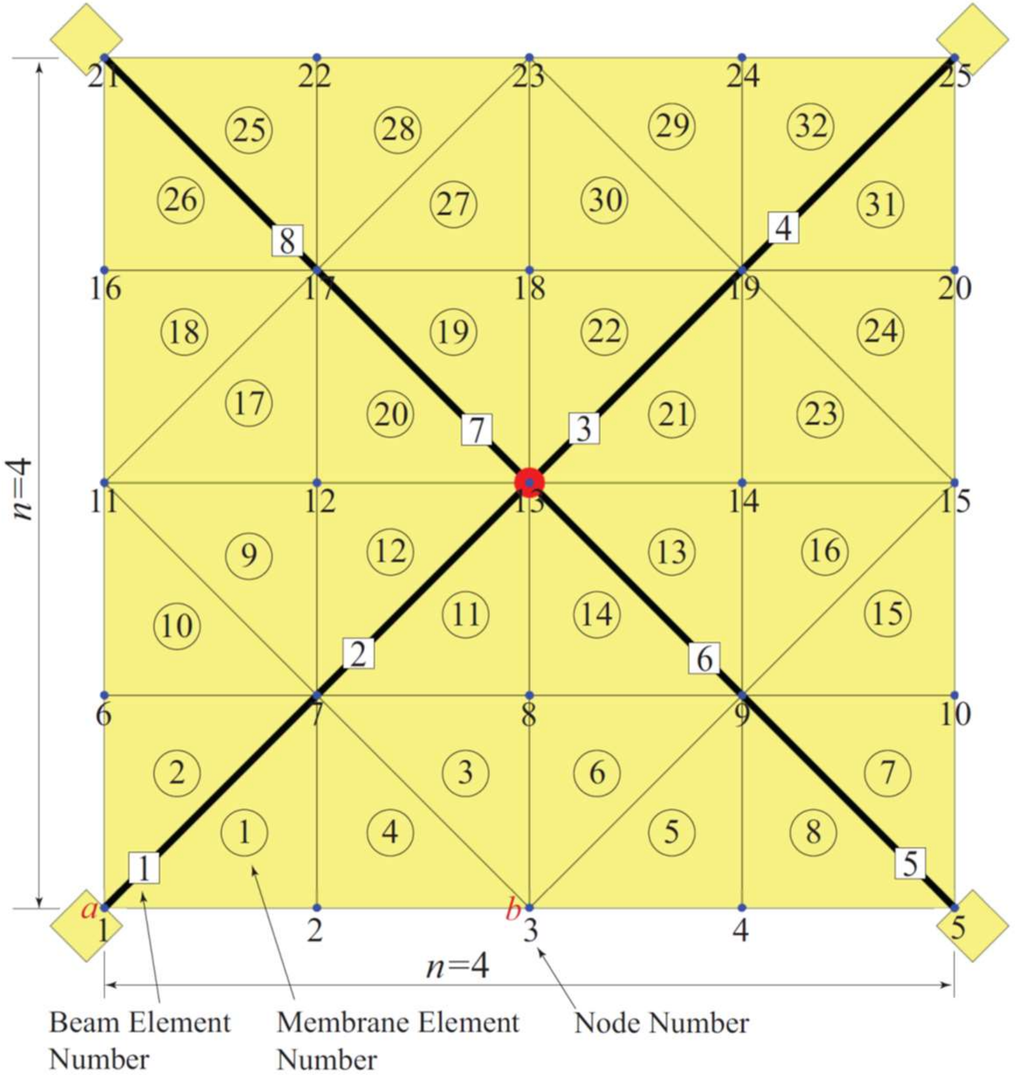

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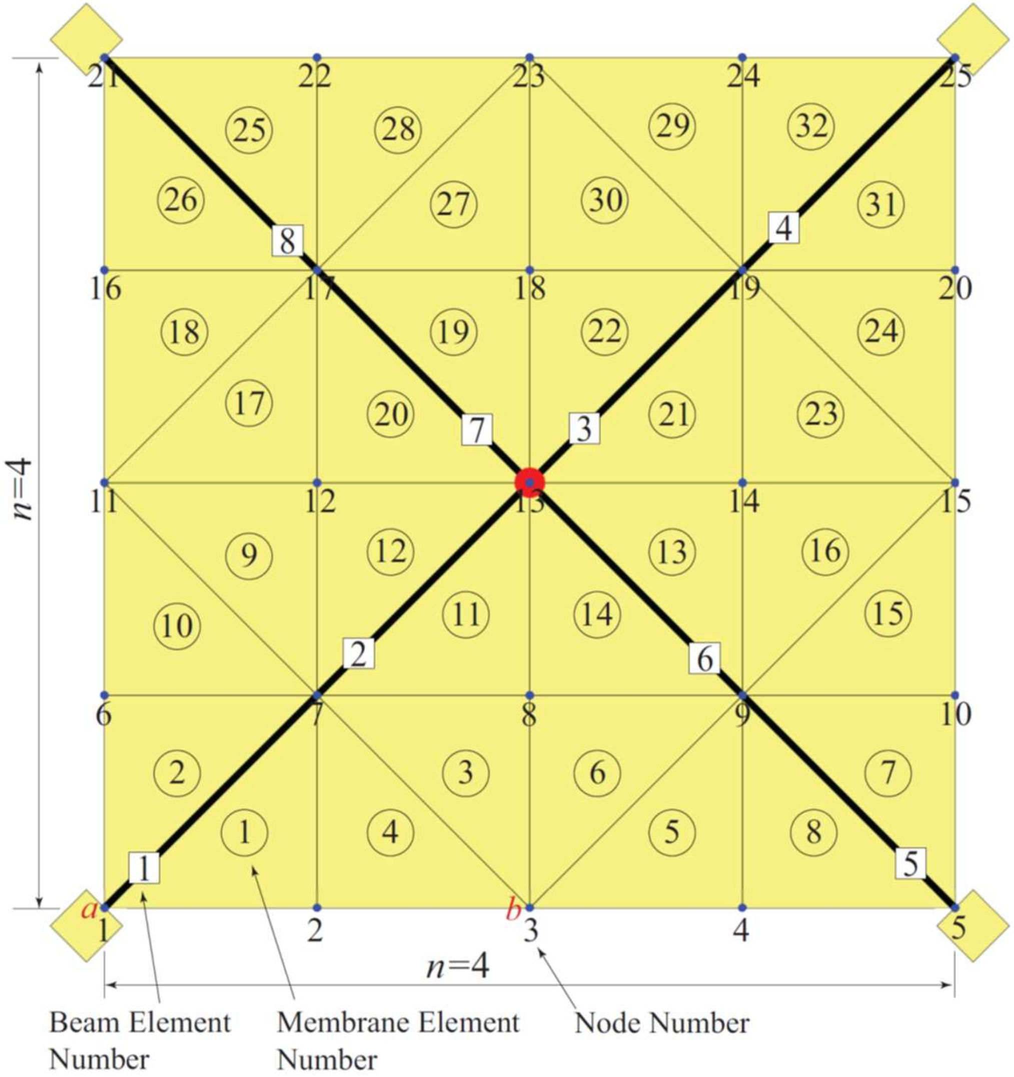
$(n+1)^2$ nodes.



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SACRAMENTO STATE
Redefine the Possible

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Both K and M are $(n^2 + 4n + 6) \times (n^2 + 4n + 6)$.

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Speed of light

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Unit normal vector to ith element

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The solar radiation force over the control vanes are

$$\mathbf{f}_c^{(i)} = 2\frac{S_0}{c}\left[\frac{R_0}{R_c^{(i)}}\right]^2 l^2 \left[\hat{\mathbf{R}}_c^{(i)} \cdot \hat{\mathbf{n}}_c^{(i)}\right]^2 \hat{\mathbf{n}}_c^{(i)}, \quad i = 1, 2, 3, 4$$

EQUATIONS OF MOTION

The total virtual work is

$$\delta W = \delta W_m + \delta W_b + \delta W_c + \delta W_p = \mathbf{F}^T \delta \mathbf{R}^* + \mathbf{M}^T \delta \boldsymbol{\theta}^* + \mathbf{Q}^T \delta \mathbf{q}$$

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SACRAMENTO STATE
Redefine the Possible

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Due to the nature of FEM, a large n must be used for sufficiently accurate representation of the system.

EQUATIONS OF MOTION

The system is also underactuated because the number of control inputs is many times smaller than the number of degrees of freedom.

PERTURBATION SOLUTION

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$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{f}\left[\bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t)\right]$$

for the desired nominal dynamics and the *first-order equation*

$$\dot{\hat{\mathbf{x}}}(t) = A\left[\bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t)\right]\hat{\mathbf{x}}(t) + B\left[\bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t)\right]\hat{\mathbf{u}}(t)$$

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Coefficient matrices, both functions of $\bar{\mathbf{x}}(t)$ and $\bar{\mathbf{u}}(t)$

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We note that the zero-order state $\bar{\mathbf{x}}(t)$ and control input $\bar{\mathbf{u}}(t)$ enter into the first-order equation as inputs.

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It can also be used to design feedback control to attenuate perturbations.

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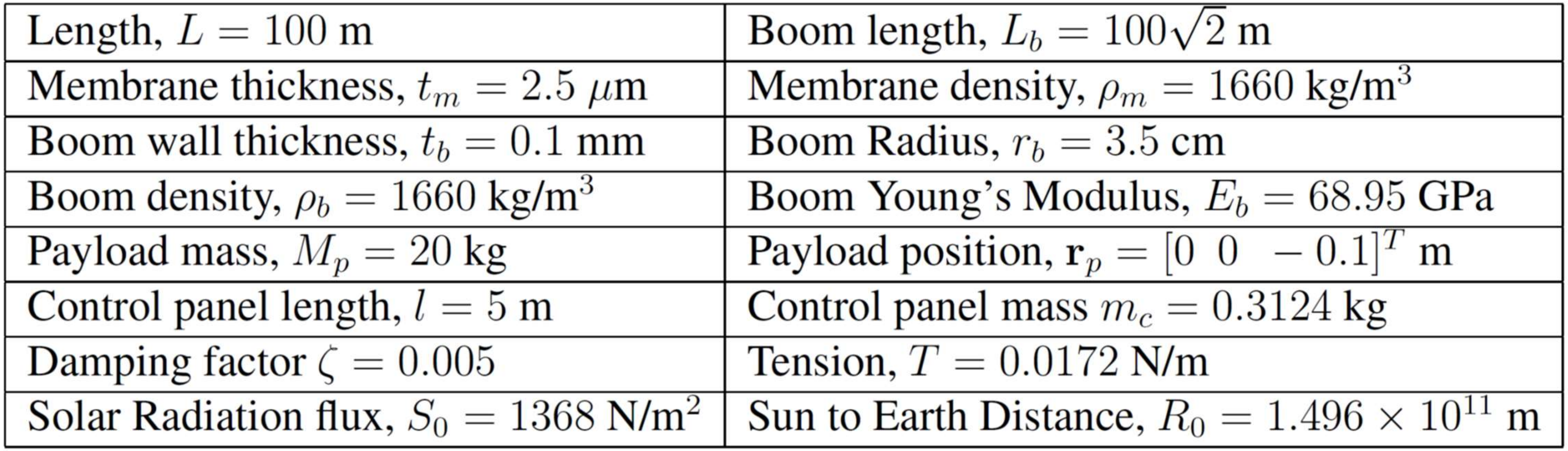
Vector of nodal displacements

where U_e is the matrix of k vibration modes and

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{bmatrix}, \quad \boldsymbol{\eta} = \dot{\boldsymbol{\xi}} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_k \end{bmatrix}$$

NUMERICAL APPLICATION

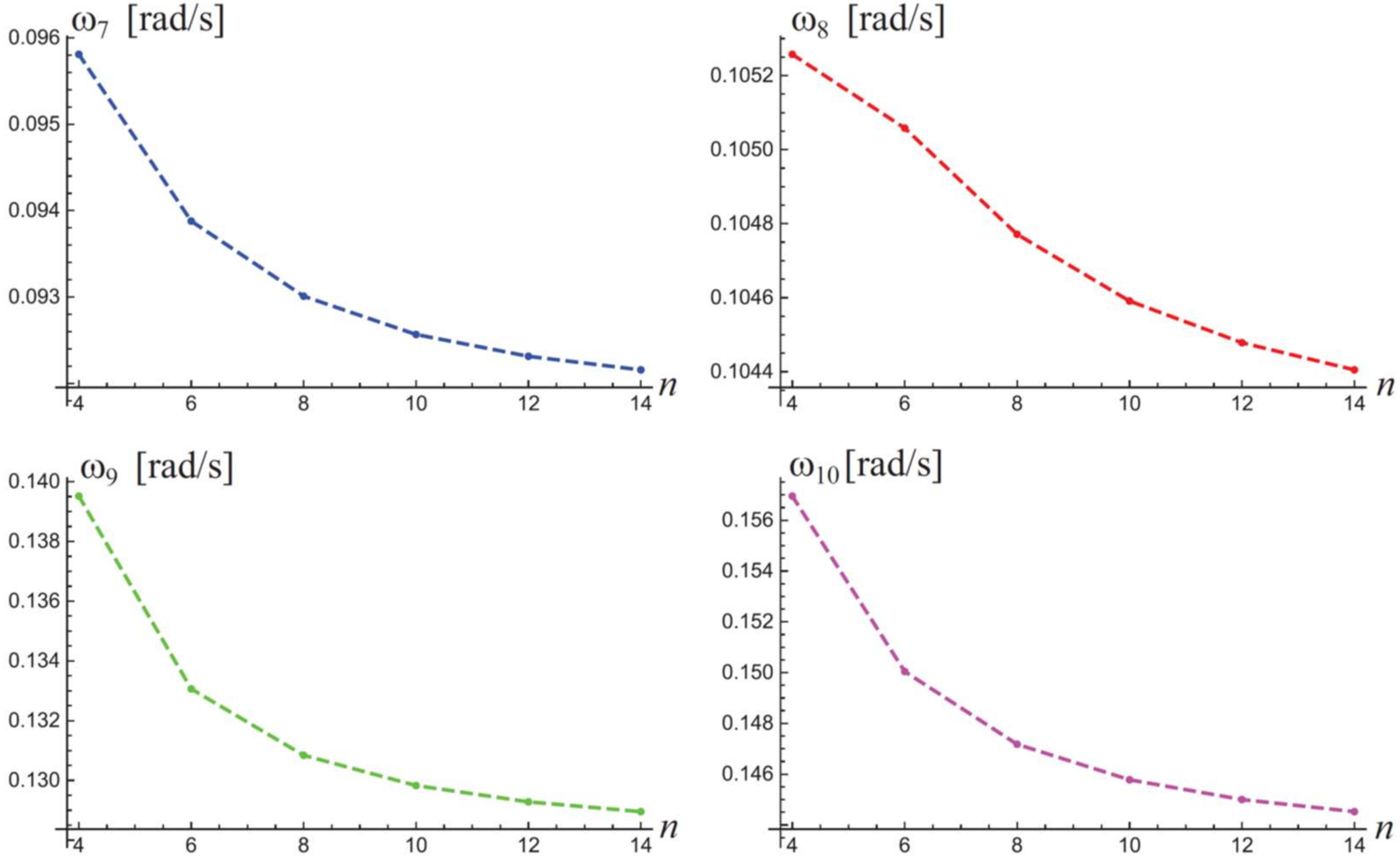
Numerical Data



Length, $L = 100$ m	Boom length, $L_b = 100\sqrt{2}$ m
Membrane thickness, $t_m = 2.5\ \mu$m	Membrane density, $\rho_m = 1660$ kg/m^3
Boom wall thickness, $t_b = 0.1$ mm	Boom Radius, $r_b = 3.5$ cm
Boom density, $\rho_b = 1660$ kg/m^3	Boom Young's Modulus, $E_b = 68.95$ GPa
Payload mass, $M_p = 20$ kg	Payload position, $\mathbf{r}_p = [0\ \ 0\ \ -0.1]^T$ m
Control panel length, $l = 5$ m	Control panel mass $m_c = 0.3124$ kg
Damping factor $\zeta = 0.005$	Tension, $T = 0.0172$ N/m
Solar Radiation flux, $S_0 = 1368$ N/m^2	Sun to Earth Distance, $R_0 = 1.496 \times 10^{11}$ m

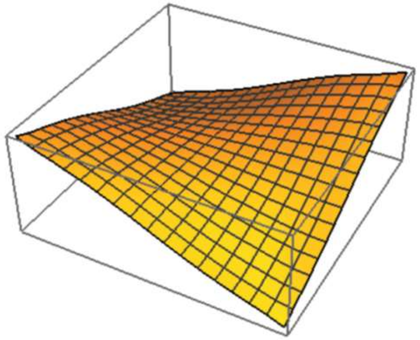
NUMERICAL APPLICATION

First 4 nonzero eigenfrequencies vs. n

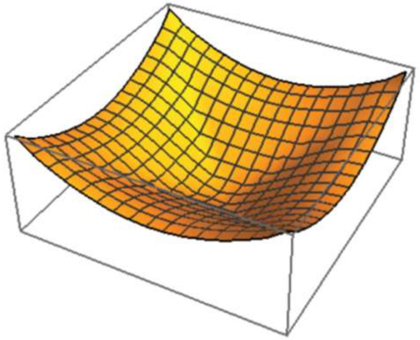


NUMERICAL APPLICATION

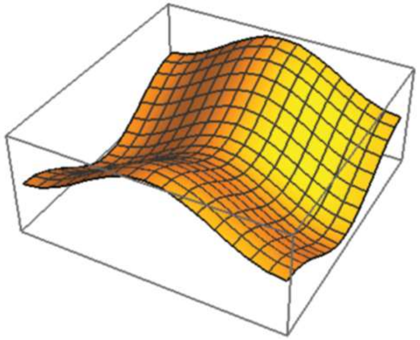
Shapes of first 8 vibration modes for $n = 14$



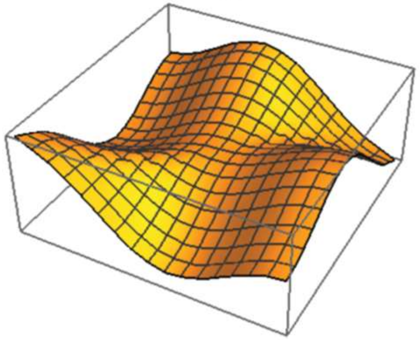
7th Mode at ω_7 =0.092156 rad/s



8th Mode at ω_8 =0.104405 rad/s



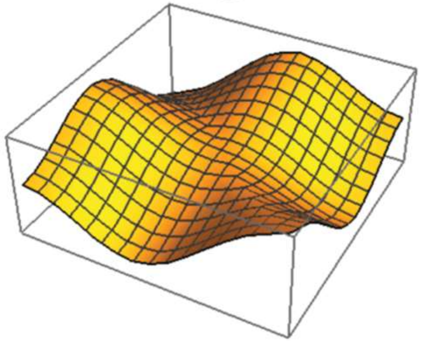
9th Mode at ω_9 =0.128955 rad/s



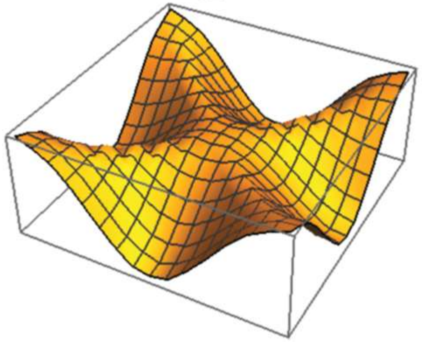
10th Mode at ω_{10}=0.144522 rad/s

NUMERICAL APPLICATION

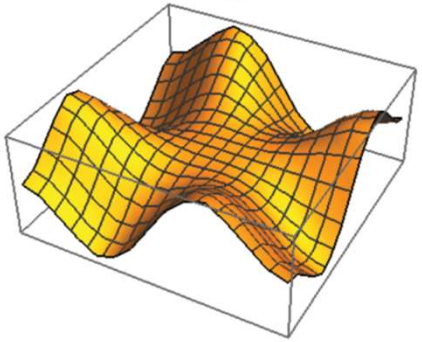
Shapes of first 8 vibration modes for $n = 14$ (continued)



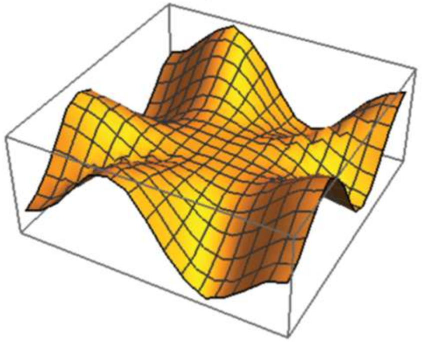
11th Mode at ω_{11}=0.144522 rad/s



12th Mode at ω_{12}=0.164341 rad/s



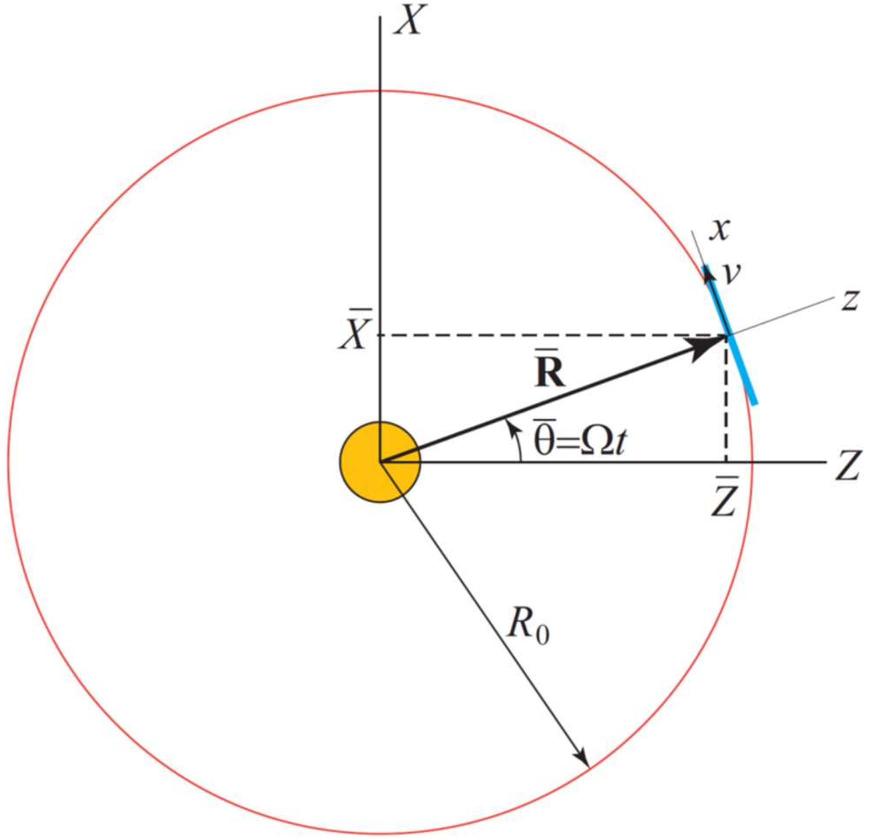
13th Mode at ω_{13}=0.2076808 rad/s



14th Mode at ω_{14}=0.220341 rad/s

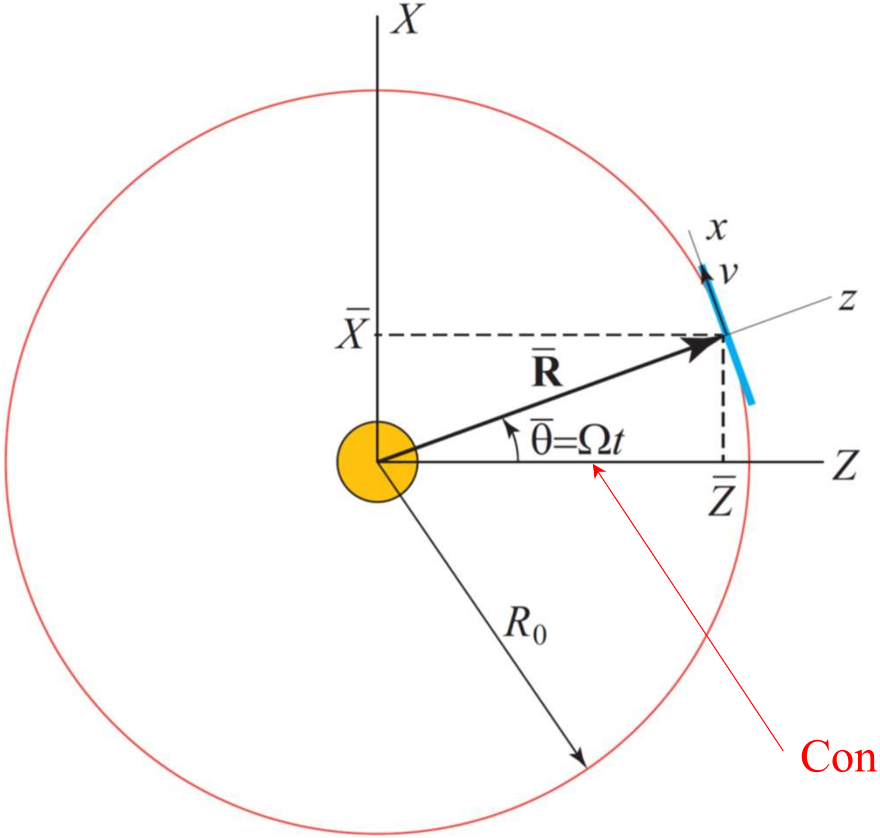
NUMERICAL APPLICATION

Solarcraft in a circular orbit around the Sun at $R = R_0 = 1$ AU



NUMERICAL APPLICATION

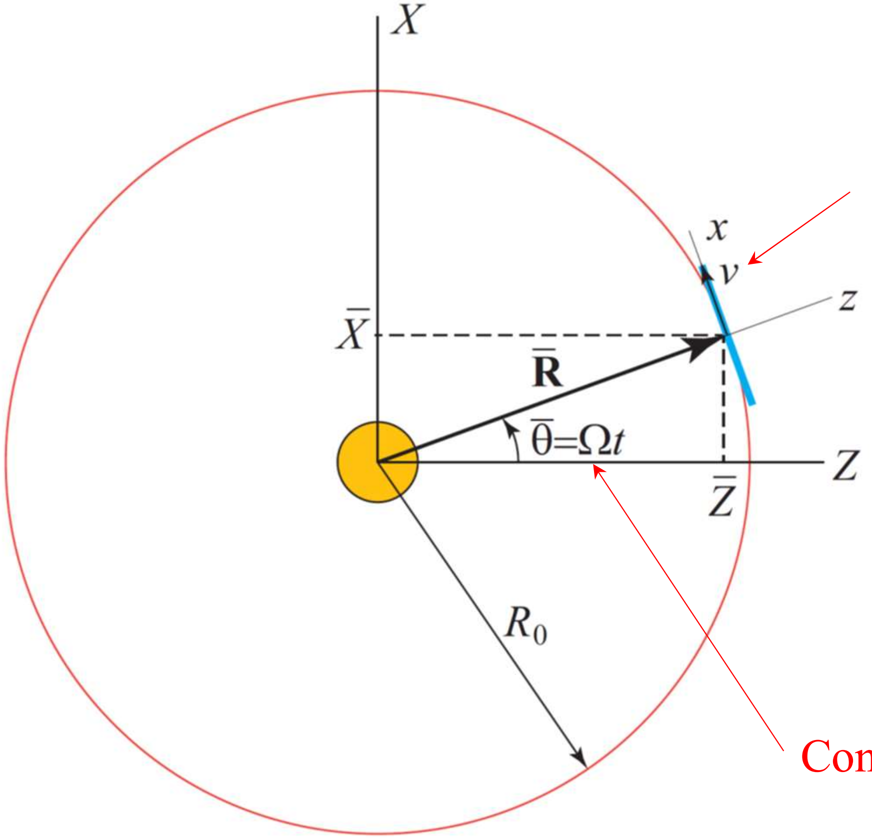
Solarcraft in a circular orbit around the Sun at $R = R_0 = 1$ AU



Constant turn rate

NUMERICAL APPLICATION

Solarcraft in a circular orbit around the Sun at $R = R_0 = 1$ AU



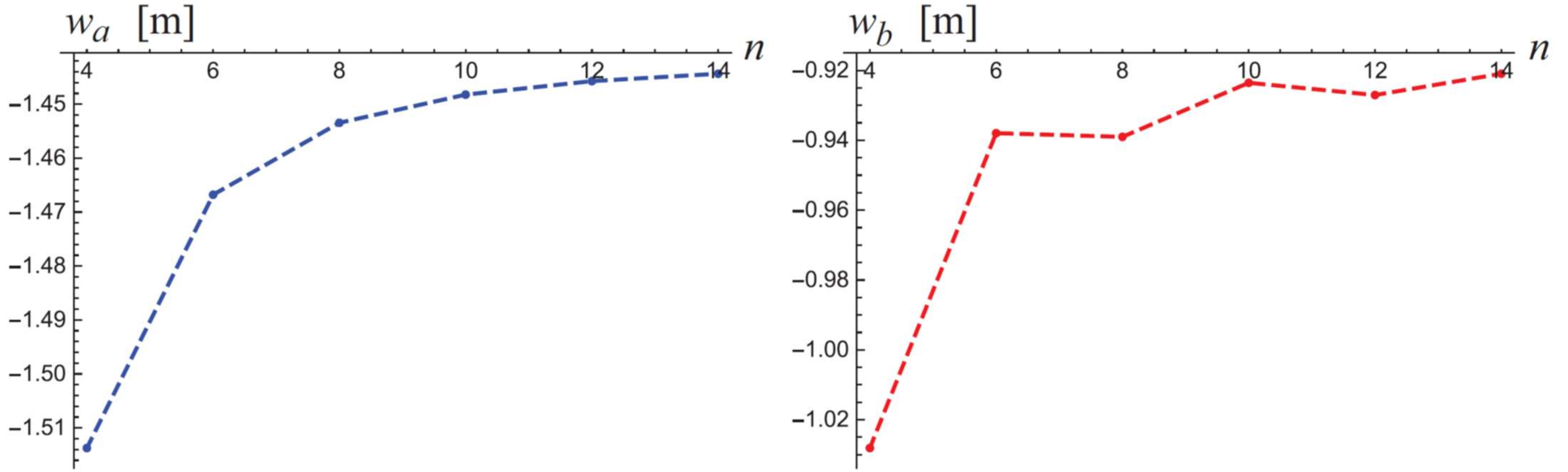
$v = R\Omega = \text{constant}$

Constant turn rate

SACRAMENTO STATE
Redefine the Possible

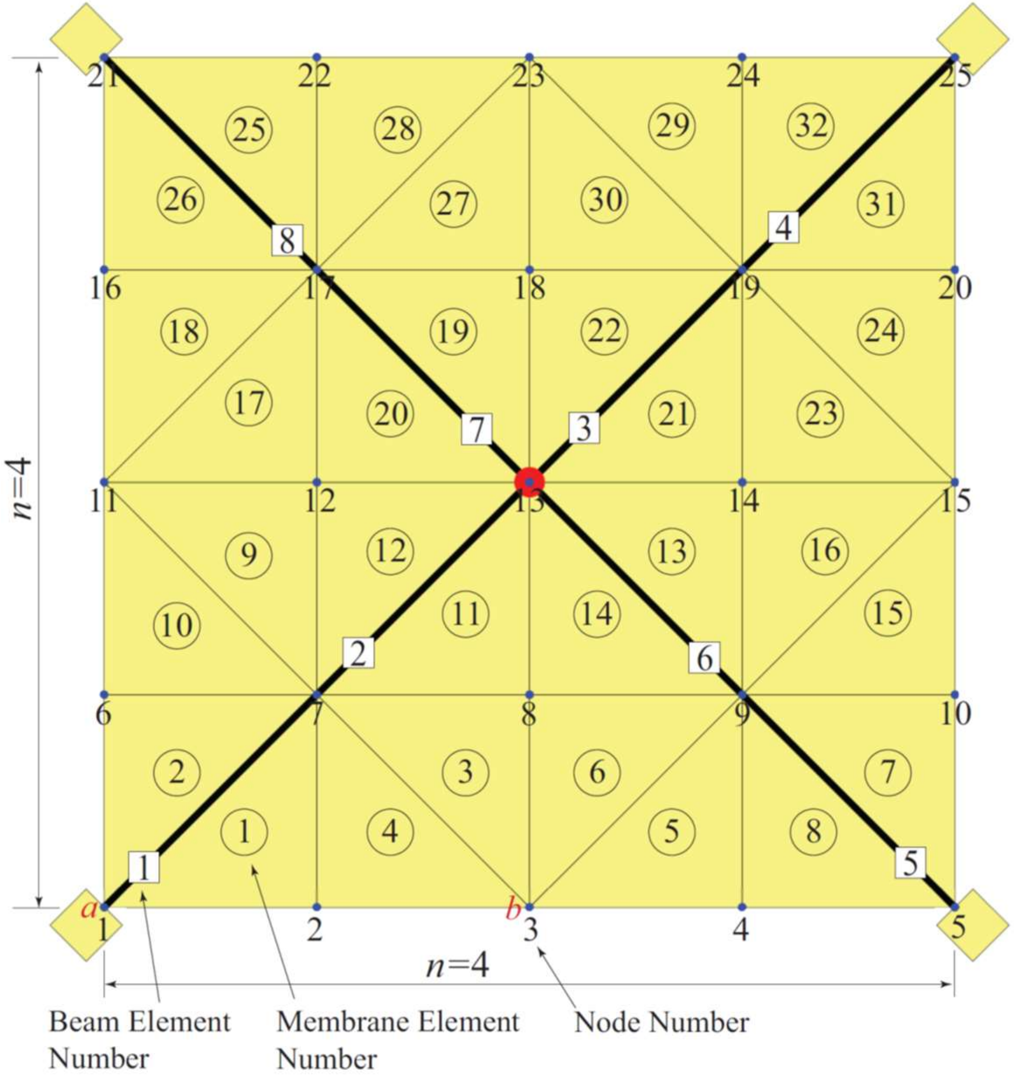
NUMERICAL APPLICATION

Convergence of the elastic displacements at the points a and b



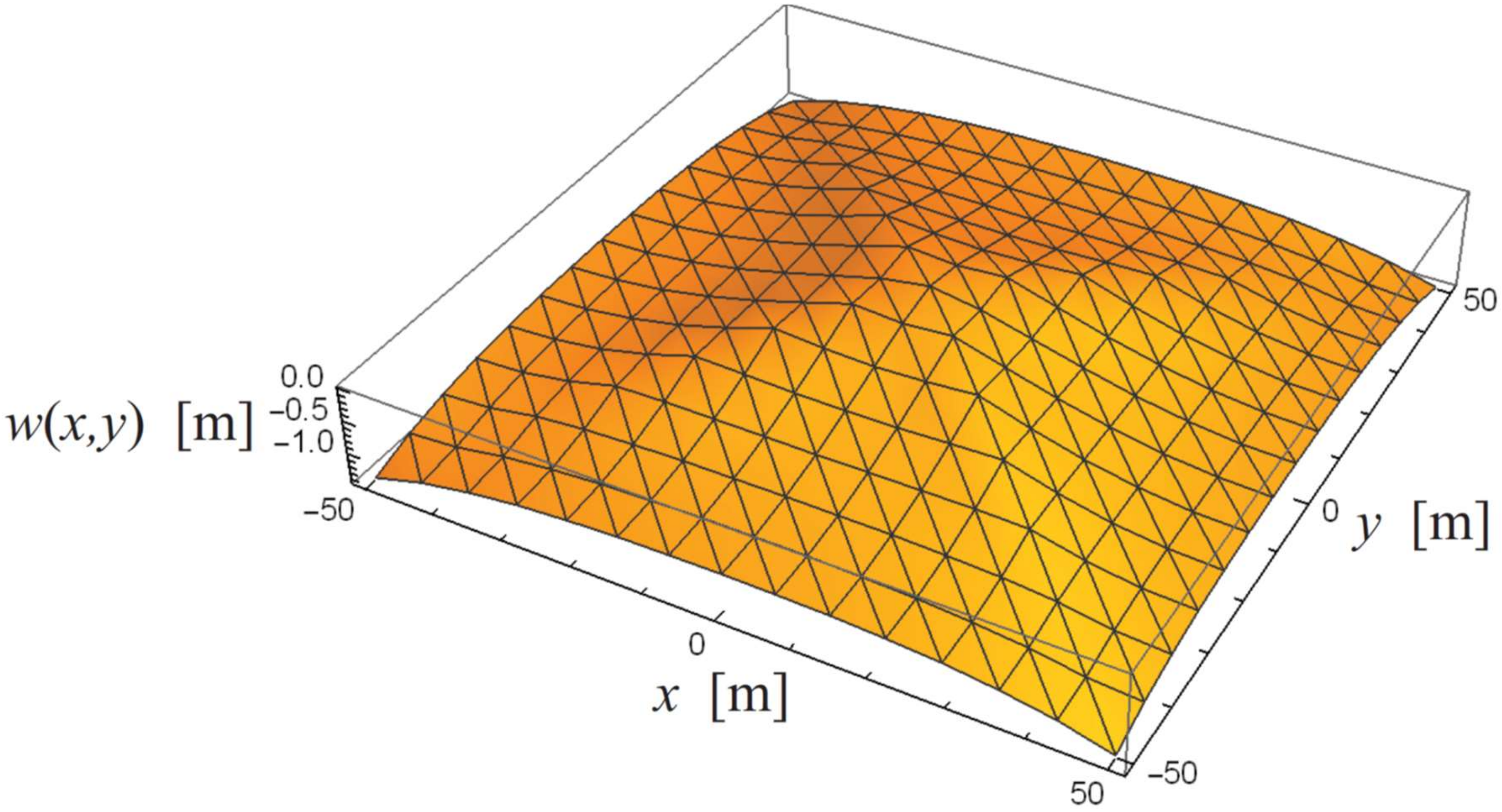
SACRAMENTO STATE
Redefine the Possible

NUMERICAL APPLICATION



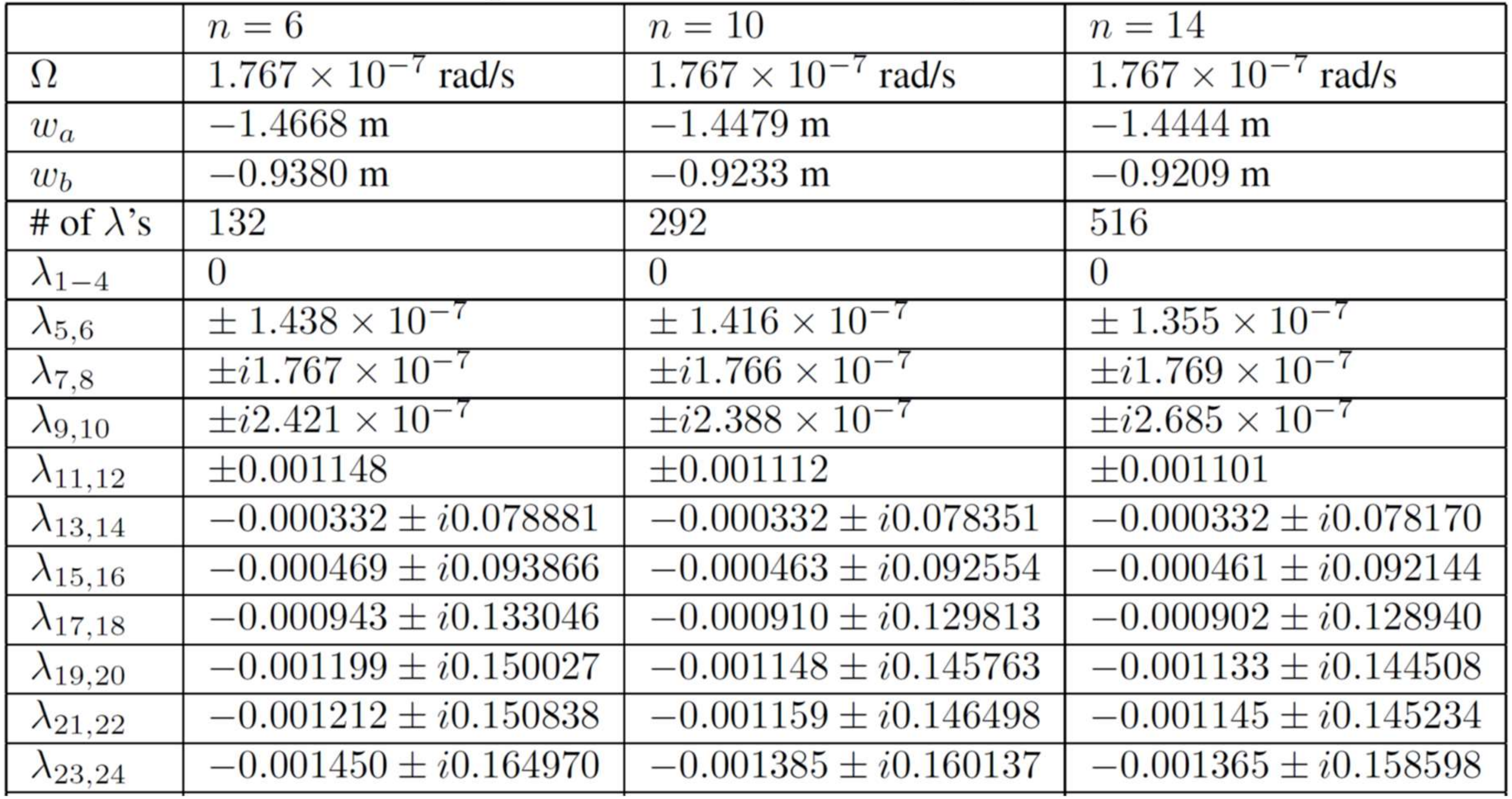
NUMERICAL APPLICATION

Elastic shape for $n = 14$ during the circular orbit



NUMERICAL APPLICATION

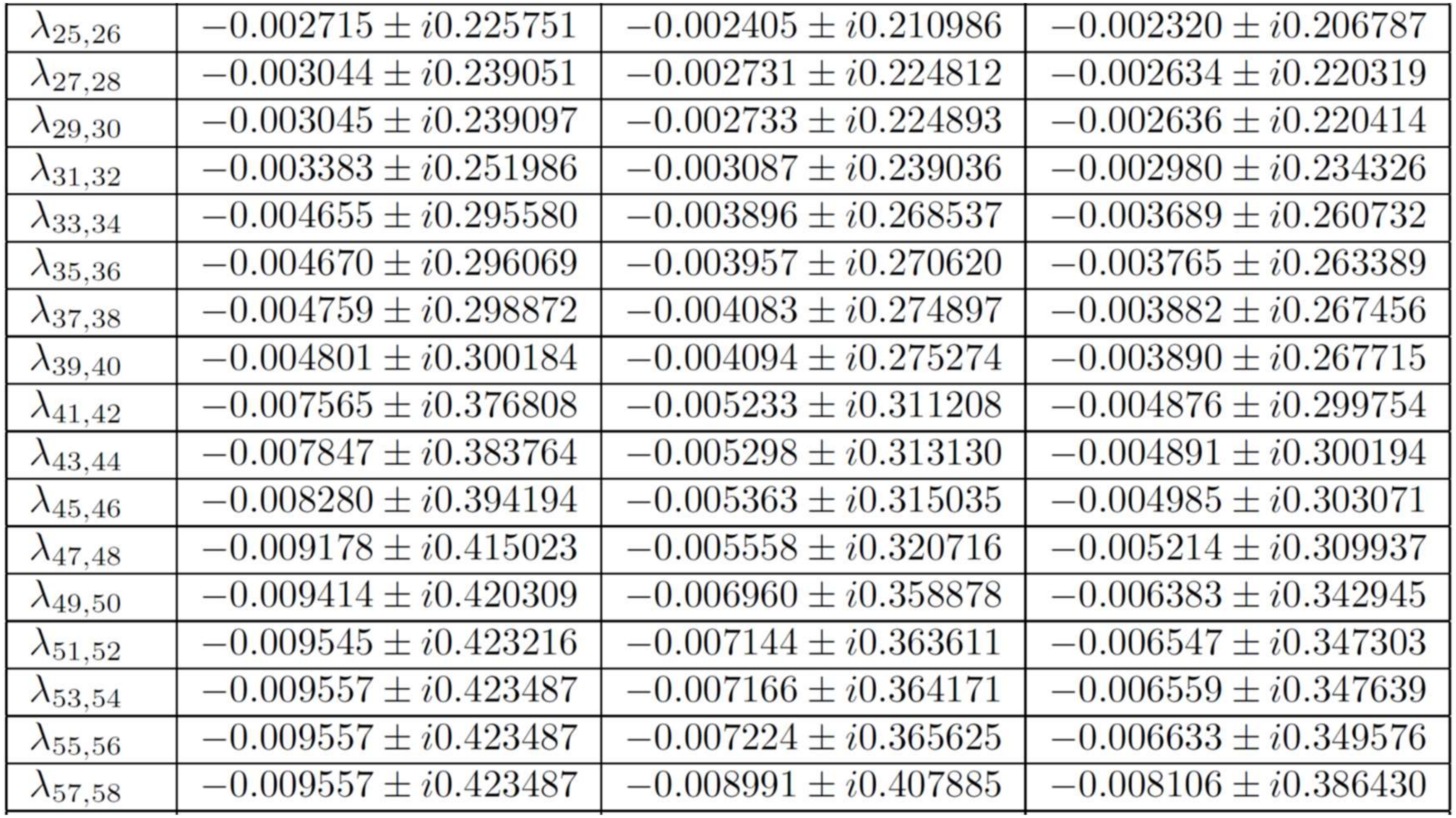
Equilibrium Values and Eigenvalues of *A* for the circular orbit at 1 AU



	$n = 6$	$n = 10$	$n = 14$
Ω	1.767×10^{-7} rad/s	1.767×10^{-7} rad/s	1.767×10^{-7} rad/s
w_a	-1.4668 m	-1.4479 m	-1.4444 m
w_b	-0.9380 m	-0.9233 m	-0.9209 m
# of λ's	132	292	516
λ_{1-4}	0	0	0
$\lambda_{5,6}$	$\pm\, 1.438 \times 10^{-7}$	$\pm\, 1.416 \times 10^{-7}$	$\pm\, 1.355 \times 10^{-7}$
$\lambda_{7,8}$	$\pm i1.767 \times 10^{-7}$	$\pm i1.766 \times 10^{-7}$	$\pm i1.769 \times 10^{-7}$
$\lambda_{9,10}$	$\pm i2.421 \times 10^{-7}$	$\pm i2.388 \times 10^{-7}$	$\pm i2.685 \times 10^{-7}$
$\lambda_{11,12}$	± 0.001148	± 0.001112	± 0.001101
$\lambda_{13,14}$	$-0.000332 \pm i0.078881$	$-0.000332 \pm i0.078351$	$-0.000332 \pm i0.078170$
$\lambda_{15,16}$	$-0.000469 \pm i0.093866$	$-0.000463 \pm i0.092554$	$-0.000461 \pm i0.092144$
$\lambda_{17,18}$	$-0.000943 \pm i0.133046$	$-0.000910 \pm i0.129813$	$-0.000902 \pm i0.128940$
$\lambda_{19,20}$	$-0.001199 \pm i0.150027$	$-0.001148 \pm i0.145763$	$-0.001133 \pm i0.144508$
$\lambda_{21,22}$	$-0.001212 \pm i0.150838$	$-0.001159 \pm i0.146498$	$-0.001145 \pm i0.145234$
$\lambda_{23,24}$	$-0.001450 \pm i0.164970$	$-0.001385 \pm i0.160137$	$-0.001365 \pm i0.158598$

NUMERICAL APPLICATION

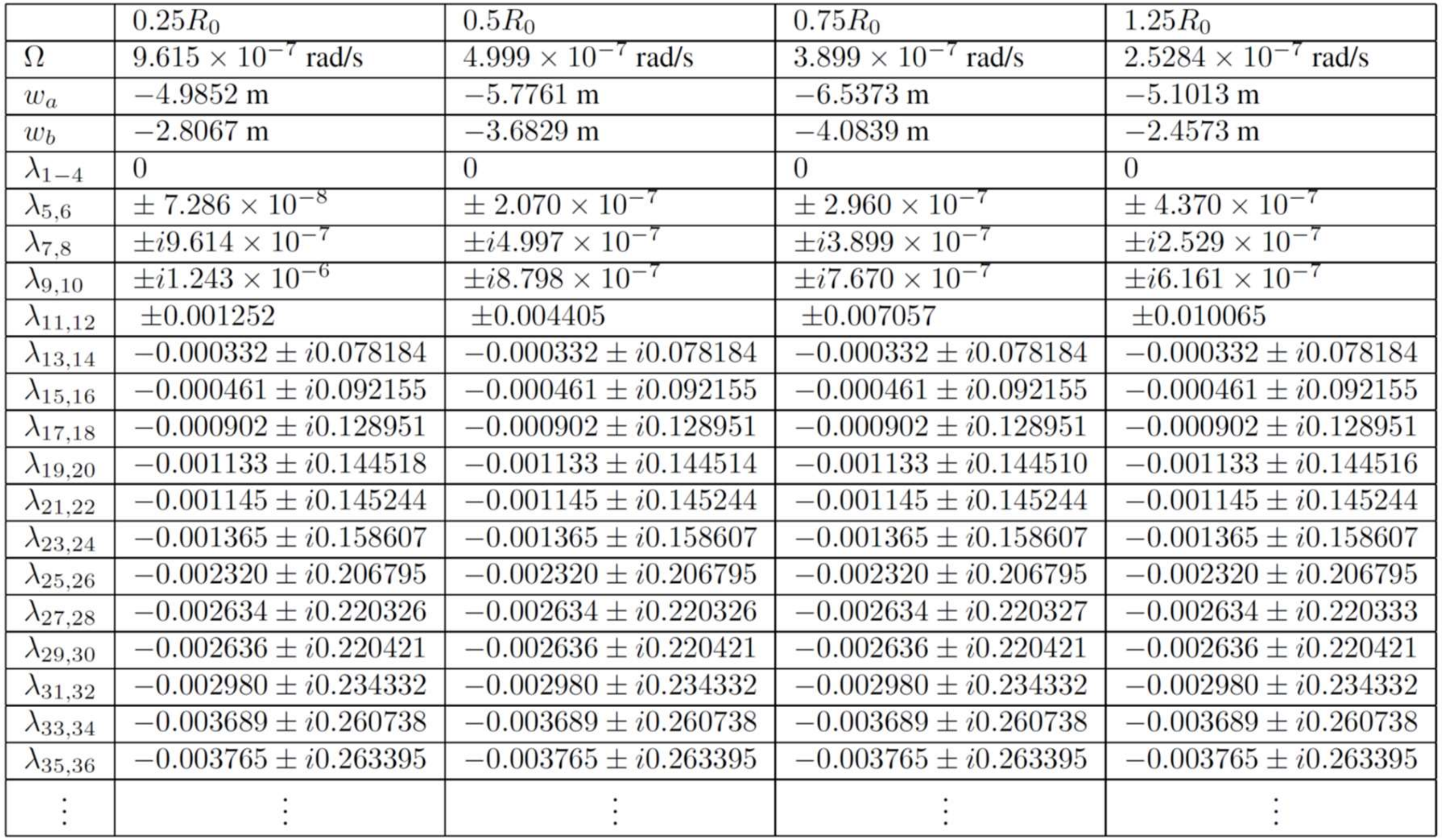
Equilibrium Values and Eigenvalues of A for the circular orbit at 1 AU



$\lambda_{25,26}$	$-0.002715 \pm i0.225751$	$-0.002405 \pm i0.210986$	$-0.002320 \pm i0.206787$
$\lambda_{27,28}$	$-0.003044 \pm i0.239051$	$-0.002731 \pm i0.224812$	$-0.002634 \pm i0.220319$
$\lambda_{29,30}$	$-0.003045 \pm i0.239097$	$-0.002733 \pm i0.224893$	$-0.002636 \pm i0.220414$
$\lambda_{31,32}$	$-0.003383 \pm i0.251986$	$-0.003087 \pm i0.239036$	$-0.002980 \pm i0.234326$
$\lambda_{33,34}$	$-0.004655 \pm i0.295580$	$-0.003896 \pm i0.268537$	$-0.003689 \pm i0.260732$
$\lambda_{35,36}$	$-0.004670 \pm i0.296069$	$-0.003957 \pm i0.270620$	$-0.003765 \pm i0.263389$
$\lambda_{37,38}$	$-0.004759 \pm i0.298872$	$-0.004083 \pm i0.274897$	$-0.003882 \pm i0.267456$
$\lambda_{39,40}$	$-0.004801 \pm i0.300184$	$-0.004094 \pm i0.275274$	$-0.003890 \pm i0.267715$
$\lambda_{41,42}$	$-0.007565 \pm i0.376808$	$-0.005233 \pm i0.311208$	$-0.004876 \pm i0.299754$
$\lambda_{43,44}$	$-0.007847 \pm i0.383764$	$-0.005298 \pm i0.313130$	$-0.004891 \pm i0.300194$
$\lambda_{45,46}$	$-0.008280 \pm i0.394194$	$-0.005363 \pm i0.315035$	$-0.004985 \pm i0.303071$
$\lambda_{47,48}$	$-0.009178 \pm i0.415023$	$-0.005558 \pm i0.320716$	$-0.005214 \pm i0.309937$
$\lambda_{49,50}$	$-0.009414 \pm i0.420309$	$-0.006960 \pm i0.358878$	$-0.006383 \pm i0.342945$
$\lambda_{51,52}$	$-0.009545 \pm i0.423216$	$-0.007144 \pm i0.363611$	$-0.006547 \pm i0.347303$
$\lambda_{53,54}$	$-0.009557 \pm i0.423487$	$-0.007166 \pm i0.364171$	$-0.006559 \pm i0.347639$
$\lambda_{55,56}$	$-0.009557 \pm i0.423487$	$-0.007224 \pm i0.365625$	$-0.006633 \pm i0.349576$
$\lambda_{57,58}$	$-0.009557 \pm i0.423487$	$-0.008991 \pm i0.407885$	$-0.008106 \pm i0.386430$

NUMERICAL APPLICATION

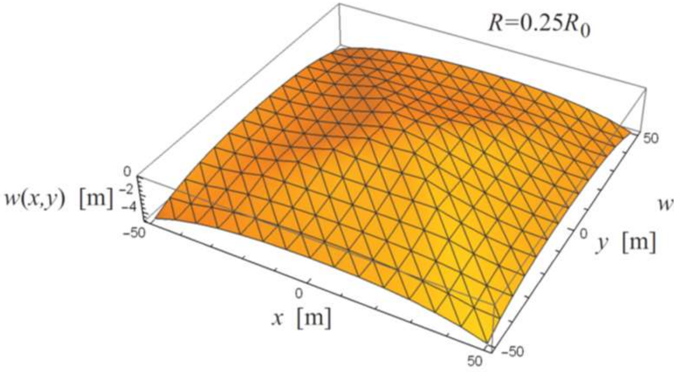
Equilibrium Values and Eigenvalues of A for $n = 14$

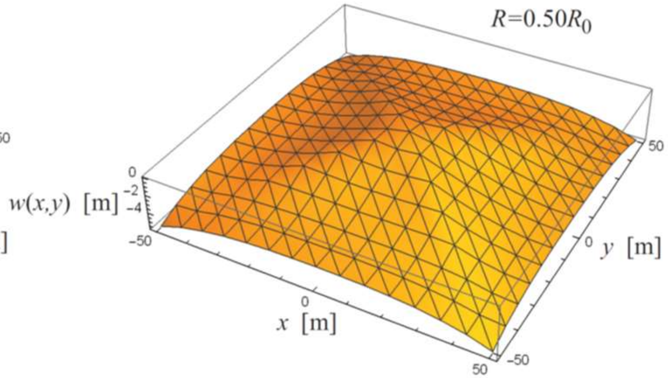


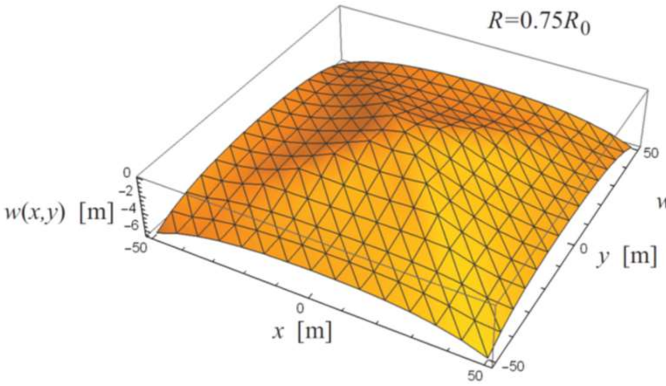
	$0.25R_0$	$0.5R_0$	$0.75R_0$	$1.25R_0$
Ω	9.615×10^{-7} rad/s	4.999×10^{-7} rad/s	3.899×10^{-7} rad/s	2.5284×10^{-7} rad/s
w_a	-4.9852 m	-5.7761 m	-6.5373 m	-5.1013 m
w_b	-2.8067 m	-3.6829 m	-4.0839 m	-2.4573 m
λ_{1-4}	0	0	0	0
$\lambda_{5,6}$	$\pm\, 7.286 \times 10^{-8}$	$\pm\, 2.070 \times 10^{-7}$	$\pm\, 2.960 \times 10^{-7}$	$\pm\, 4.370 \times 10^{-7}$
$\lambda_{7,8}$	$\pm i9.614 \times 10^{-7}$	$\pm i4.997 \times 10^{-7}$	$\pm i3.899 \times 10^{-7}$	$\pm i2.529 \times 10^{-7}$
$\lambda_{9,10}$	$\pm i1.243 \times 10^{-6}$	$\pm i8.798 \times 10^{-7}$	$\pm i7.670 \times 10^{-7}$	$\pm i6.161 \times 10^{-7}$
$\lambda_{11,12}$	± 0.001252	± 0.004405	± 0.007057	± 0.010065
$\lambda_{13,14}$	$-0.000332 \pm i0.078184$	$-0.000332 \pm i0.078184$	$-0.000332 \pm i0.078184$	$-0.000332 \pm i0.078184$
$\lambda_{15,16}$	$-0.000461 \pm i0.092155$	$-0.000461 \pm i0.092155$	$-0.000461 \pm i0.092155$	$-0.000461 \pm i0.092155$
$\lambda_{17,18}$	$-0.000902 \pm i0.128951$	$-0.000902 \pm i0.128951$	$-0.000902 \pm i0.128951$	$-0.000902 \pm i0.128951$
$\lambda_{19,20}$	$-0.001133 \pm i0.144518$	$-0.001133 \pm i0.144514$	$-0.001133 \pm i0.144510$	$-0.001133 \pm i0.144516$
$\lambda_{21,22}$	$-0.001145 \pm i0.145244$	$-0.001145 \pm i0.145244$	$-0.001145 \pm i0.145244$	$-0.001145 \pm i0.145244$
$\lambda_{23,24}$	$-0.001365 \pm i0.158607$	$-0.001365 \pm i0.158607$	$-0.001365 \pm i0.158607$	$-0.001365 \pm i0.158607$
$\lambda_{25,26}$	$-0.002320 \pm i0.206795$	$-0.002320 \pm i0.206795$	$-0.002320 \pm i0.206795$	$-0.002320 \pm i0.206795$
$\lambda_{27,28}$	$-0.002634 \pm i0.220326$	$-0.002634 \pm i0.220326$	$-0.002634 \pm i0.220327$	$-0.002634 \pm i0.220333$
$\lambda_{29,30}$	$-0.002636 \pm i0.220421$	$-0.002636 \pm i0.220421$	$-0.002636 \pm i0.220421$	$-0.002636 \pm i0.220421$
$\lambda_{31,32}$	$-0.002980 \pm i0.234332$	$-0.002980 \pm i0.234332$	$-0.002980 \pm i0.234332$	$-0.002980 \pm i0.234332$
$\lambda_{33,34}$	$-0.003689 \pm i0.260738$	$-0.003689 \pm i0.260738$	$-0.003689 \pm i0.260738$	$-0.003689 \pm i0.260738$
$\lambda_{35,36}$	$-0.003765 \pm i0.263395$	$-0.003765 \pm i0.263395$	$-0.003765 \pm i0.263395$	$-0.003765 \pm i0.263395$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

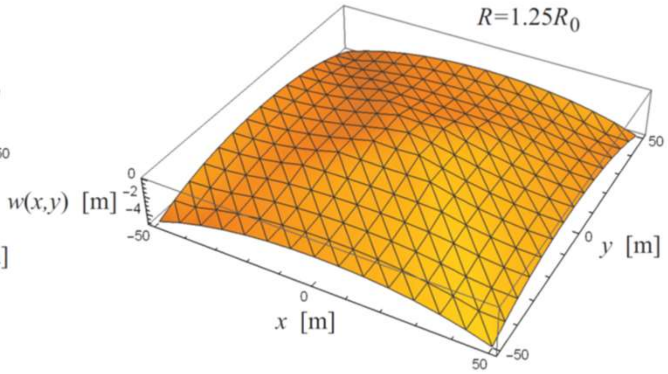
NUMERICAL APPLICATION

Elastic shapes for $n = 14$



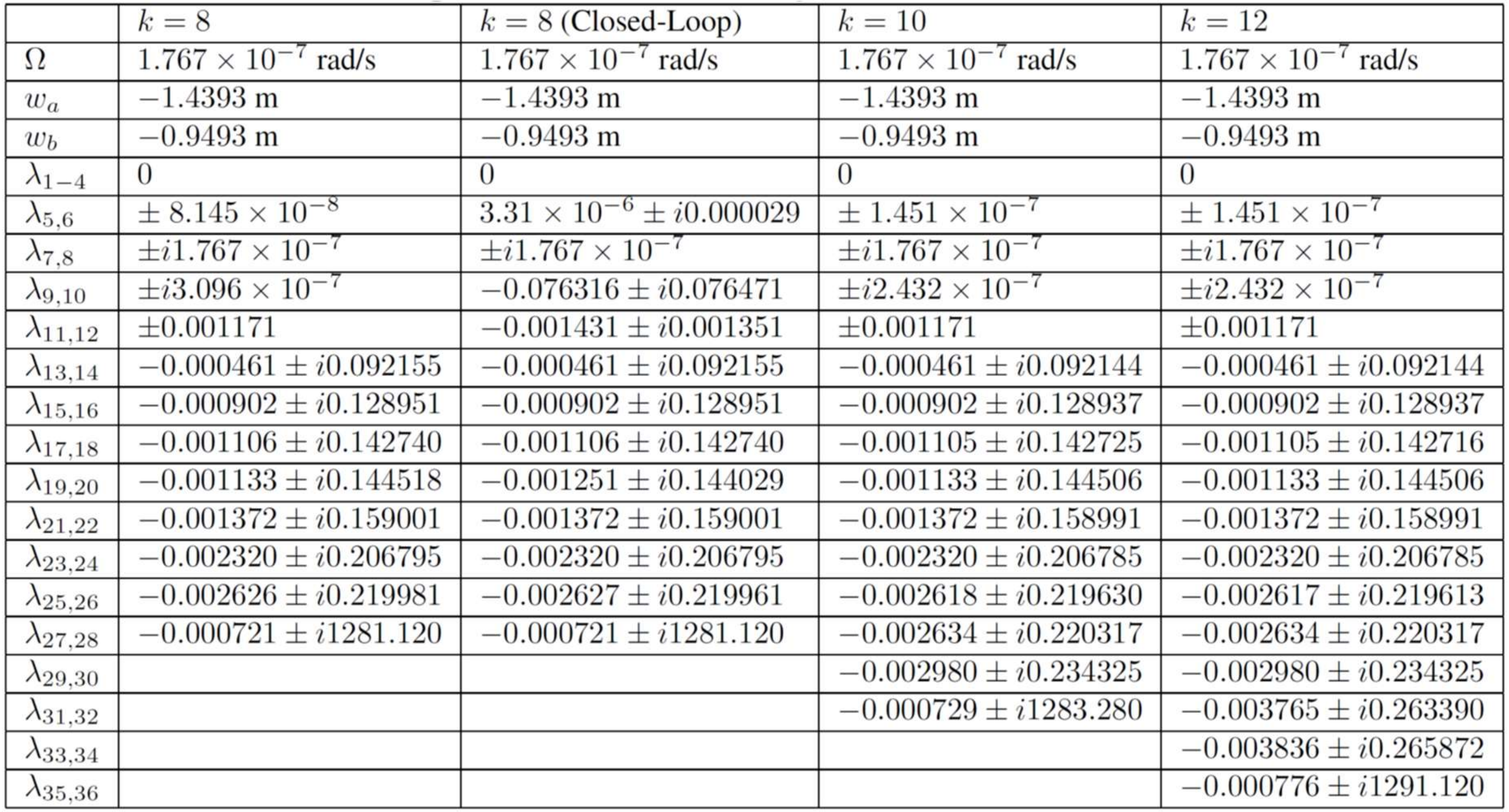






NUMERICAL APPLICATION

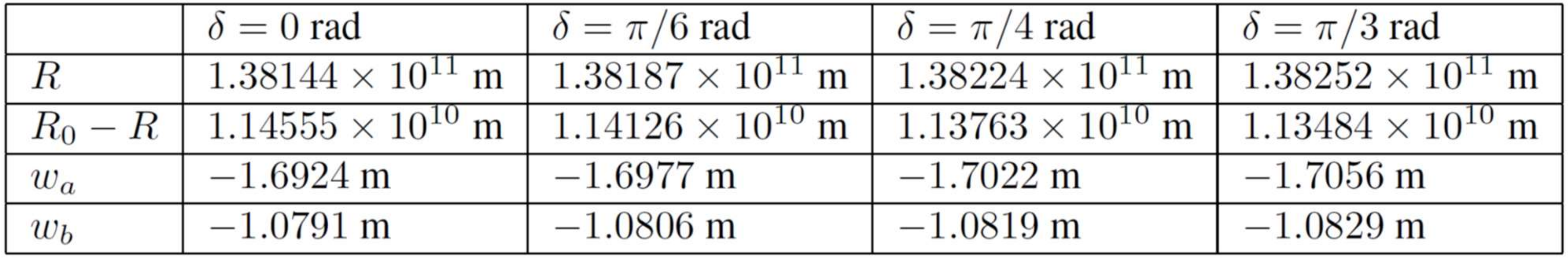
Equilibrium Values and Eigenvalues of A_r for $n = 14$



	$k = 8$	$k = 8$ (Closed-Loop)	$k = 10$	$k = 12$
Ω	1.767×10^{-7} rad/s	1.767×10^{-7} rad/s	1.767×10^{-7} rad/s	1.767×10^{-7} rad/s
w_a	-1.4393 m	-1.4393 m	-1.4393 m	-1.4393 m
w_b	-0.9493 m	-0.9493 m	-0.9493 m	-0.9493 m
λ_{1-4}	0	0	0	0
$\lambda_{5,6}$	$\pm\ 8.145 \times 10^{-8}$	$3.31 \times 10^{-6} \pm i0.000029$	$\pm\ 1.451 \times 10^{-7}$	$\pm\ 1.451 \times 10^{-7}$
$\lambda_{7,8}$	$\pm i1.767 \times 10^{-7}$	$\pm i1.767 \times 10^{-7}$	$\pm i1.767 \times 10^{-7}$	$\pm i1.767 \times 10^{-7}$
$\lambda_{9,10}$	$\pm i3.096 \times 10^{-7}$	$-0.076316 \pm i0.076471$	$\pm i2.432 \times 10^{-7}$	$\pm i2.432 \times 10^{-7}$
$\lambda_{11,12}$	± 0.001171	$-0.001431 \pm i0.001351$	± 0.001171	± 0.001171
$\lambda_{13,14}$	$-0.000461 \pm i0.092155$	$-0.000461 \pm i0.092155$	$-0.000461 \pm i0.092144$	$-0.000461 \pm i0.092144$
$\lambda_{15,16}$	$-0.000902 \pm i0.128951$	$-0.000902 \pm i0.128951$	$-0.000902 \pm i0.128937$	$-0.000902 \pm i0.128937$
$\lambda_{17,18}$	$-0.001106 \pm i0.142740$	$-0.001106 \pm i0.142740$	$-0.001105 \pm i0.142725$	$-0.001105 \pm i0.142716$
$\lambda_{19,20}$	$-0.001133 \pm i0.144518$	$-0.001251 \pm i0.144029$	$-0.001133 \pm i0.144506$	$-0.001133 \pm i0.144506$
$\lambda_{21,22}$	$-0.001372 \pm i0.159001$	$-0.001372 \pm i0.159001$	$-0.001372 \pm i0.158991$	$-0.001372 \pm i0.158991$
$\lambda_{23,24}$	$-0.002320 \pm i0.206795$	$-0.002320 \pm i0.206795$	$-0.002320 \pm i0.206785$	$-0.002320 \pm i0.206785$
$\lambda_{25,26}$	$-0.002626 \pm i0.219981$	$-0.002627 \pm i0.219961$	$-0.002618 \pm i0.219630$	$-0.002617 \pm i0.219613$
$\lambda_{27,28}$	$-0.000721 \pm i1281.120$	$-0.000721 \pm i1281.120$	$-0.002634 \pm i0.220317$	$-0.002634 \pm i0.220317$
$\lambda_{29,30}$			$-0.002980 \pm i0.234325$	$-0.002980 \pm i0.234325$
$\lambda_{31,32}$			$-0.000729 \pm i1283.280$	$-0.003765 \pm i0.263390$
$\lambda_{33,34}$				$-0.003836 \pm i0.265872$
$\lambda_{35,36}$				$-0.000776 \pm i1291.120$

NUMERICAL APPLICATION

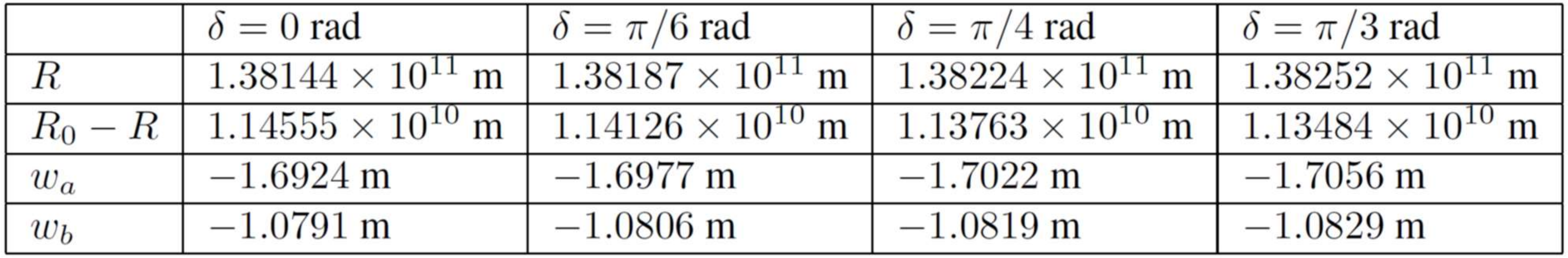
Equilibrium values for $n = 14$ at a circular orbit at the L1 libration point ($\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0,\ \delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta$)



	$\delta = 0$ rad	$\delta = \pi/6$ rad	$\delta = \pi/4$ rad	$\delta = \pi/3$ rad
R	1.38144×10^{11} m	1.38187×10^{11} m	1.38224×10^{11} m	1.38252×10^{11} m
$R_0 - R$	1.14555×10^{10} m	1.14126×10^{10} m	1.13763×10^{10} m	1.13484×10^{10} m
w_a	-1.6924 m	-1.6977 m	-1.7022 m	-1.7056 m
w_b	-1.0791 m	-1.0806 m	-1.0819 m	-1.0829 m

NUMERICAL APPLICATION

Equilibrium values for $n = 14$ at a circular orbit at the L1 libration point ($\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0,\ \delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta$)



	$\delta = 0$ rad	$\delta = \pi/6$ rad	$\delta = \pi/4$ rad	$\delta = \pi/3$ rad
R	1.38144×10^{11} m	1.38187×10^{11} m	1.38224×10^{11} m	1.38252×10^{11} m
$R_0 - R$	1.14555×10^{10} m	1.14126×10^{10} m	1.13763×10^{10} m	1.13484×10^{10} m
w_a	-1.6924 m	-1.6977 m	-1.7022 m	-1.7056 m
w_b	-1.0791 m	-1.0806 m	-1.0819 m	-1.0829 m

For a conventional spacecraft $R = 1.48118 \times 10^{11}$ m and

$R - R_0 = 1.48242 \times 10^{9}$ m.

CONCLUSIONS

This presents a comprehensive mathematical model for the dynamics of a solarcraft, accounting for both rigid body and elastic motions and their interactions.

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Nonlinearity of the governing equation is dealt with a perturbation solution that separates the equation into zero-order and first-order equations.

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This presents a comprehensive mathematical model for the dynamics of a solarcraft, accounting for both rigid body and elastic motions and their interactions.

Nonlinearity of the governing equation is dealt with a perturbation solution that separates the equation into zero-order and first-order equations.

The zero-order equation is used to design an open-loop control for a desired maneuver.

CONCLUSIONS

The first-order equation is used to assess the stability about the desired maneuver and to design feedback control to alleviate perturbations.

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High dimensionality and underactuation are dealt with a model reduction approach.

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In numerical application, circular orbits around the Sun are considered.

CONCLUSIONS

The first-order equation is used to assess the stability about the desired maneuver and to design feedback control to alleviate perturbations.

High dimensionality and underactuation are dealt with a model reduction approach.

In numerical application, circular orbits around the Sun are considered. The zero-order equation is used to determine the turn rate and elastic deformation.

CONCLUSIONS

Circular orbit at the sub-L1 libration point is also considered.

The first-order equation is used to assess the stability about the orbits.

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The first-order equation is used to assess the stability about the orbits.

Future research will focus on introduction of more effective control inputs, and control design.