

NEW YORK CITY COLLEGE OF TECHNOLOGY
CITY UNIVERSITY OF NEW YORK

Head Start to Precalculus

A Preparatory Workshop for
MAT 1375 Precalculus

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Head Start to Precalculus

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In Memory of our beloved friend and colleague

Professor Janet Liou-Mark

Whose devotion and contribution to the education of City Tech students will be remembered

This workbook is created to provide students with a review and an introduction to Precalculus before the actual Precalculus course. Studies have shown that students who attend a preparatory workshop for the course tend to perform better in the course. The Precalculus workshop bears no college credits nor contributes towards graduation requirement. It may not be used to substitute for nor exempt from the Precalculus requirement.

The Precalculus workshop meets four days, three hours a day, for a total of 12 hours during the week before the start of the semester. The workshop is facilitated by instructors and/or peer leaders.

Module I

Review of Factoring, Radicals, and Quadratic Equations

Section 1.1 Factoring Review

The Greatest Common Factor

The greatest common factor (GCF) for a polynomial is the largest monomial that divides each term of the polynomial.

Factoring the greatest common factor of a polynomial:

1. Determine the greatest common factor
2. Write the answer in factored form.

The GCF factoring process is the reverse of the Distributive Property

The Distributive Property:

Multiply

$$a(b + c) = ab + ac$$

The GCF factoring:

Factor/Divide

$$ab + ac = a(b + c)$$

Exercise 1: Factor the greatest common factor and express the answer in factored form. Check by multiplying using the Distributive Property.

a) Factor $5p^3 + 15p^2 - 30p$

b) Factor $9a^3b^4 - 6a^2b^3 + 3ab^2$

c) Factor $5(x + y) - 6x(x + y)$

Factoring by Grouping

Before factoring by grouping, first we factor out GCF from all four terms.

Steps in factoring by grouping (Assume there is no GCF)

1. Group pairs of terms and factor each pair.
2. If there is a common binomial factor, then factor it out.
3. If there is no common binomial factor, then interchange the middle two terms and repeat the process over. If there is still no common binomial, then the polynomial cannot be factored.

Factor $4x + 6y + 2xy + 3y^2$ by grouping

Step 1: Group the pairs of terms	$(4x + 6y) + (2xy + 3y^2)$
Step 2: Factor the GCF from each pair	$2(2x + 3y) + y(2x + 3y)$
Step 3: Factor the common binomial factor	$(2x + 3y)(2 + y)$

Exercise 2: Factor by grouping. Follow the steps in the table

a) Factor by grouping: $56 + 21k + 8h + 3hk$

b) Factor by grouping: $5x^2 + 40x - xy - 8y$

Factoring Trinomials

The **FOIL (Front-Outer-Inner-Last) Method:** $(a + b)(c + d) = ac + ad + bc + bd$

$$(a + b)(c + d) = ac + ad + bc + bd$$

Factoring Trinomials with Lead Coefficients of 1

Steps to factoring trinomials with lead coefficient of 1

1. Write the trinomial in descending powers.
2. List the factorizations of the constant (third) term of the trinomial.
3. Pick the factorization where the sum of the factors is the coefficient of the middle term.
4. Check by multiplying the binomials.

Exercise 6: Factor the trinomial with lead coefficient 1 by checking for the correct pair of product and sum

a) $x^2 - 8x + 12$

b) $y^2 - 9y - 36$

c) $x^2 + 14xy + 45y^2$

d) $a^2 + 7ab - 8b^2$

Factoring Trinomials with Lead Coefficients other than 1

Method 1: Trial and error

1. List the factorizations of the third term of the trinomial.
2. Write them as two binomials and determine the correct combination where the sum of the outer product, ad , and the inner product, bc , is equal to the middle term of the trinomial.

$$(a + b)(c + d) = ac + ad + bc + bd$$

Method 2: Factoring by grouping

1. Form the product ac .
2. Find a pair of numbers whose product is ac and whose sum is b .
3. Rewrite the polynomial to be factored so that the middle term bx is written as the sum of the two terms whose coefficients are the two numbers found in step 2.
4. Factor by grouping.

Exercise 7: Factor each trinomial below

a) $2h^2 - 5h - 3$

b) $2h^2 - h - 3$

c) $2h^2 - 5h - 12$

d) $3k^2 - 14k - 5$

Factoring Special Products

Difference of Two Squares: $a^2 - b^2 = (a + b)(a - b)$

Exercise 10: Factor each binomial below. Check if it is a difference of two squares.

a) $x^2 - 49$

b) $25 - b^2$

c) $36x^2 - 16y^2$

d) $100t^2 - 49r^2$

e) Factor completely $16x^4 - 81$

f) $x^2 + 36$

What have you noticed about the sum of two squares? _____

Square of a Binomial

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

Exercise 11: Factor each trinomial below. Check that the first and the third terms are perfect square, and the middle term is $2ab$, then apply the perfect square formula

a) $p^2 + 10p + 25$

b) $9b^2 + 42b + 49$

c) $16a^2 - 40a + 25$

d) $16y^2 - 72y + 81$

Sum of Two Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Difference of Two Cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Exercise 12: Factor each trinomial below. Check if they are sum or difference of two cubes

a) Factor $x^3 + 8$

b) Factor $27y^3 - 1$

c) Factor $64x^3 - 27y^3$

Cube of a binomial

$$a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$$

$$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$$

Exercise 13: Factor each polynomial below. Check if they are cube of a binomial.

a) $x^3 - 3x^2 + 3x - 1$

b) $x^3 + 6x^2 + 12x + 8$

c) $x^3 - 9x^2 + 27x - 27$

Mixed Factoring/Factoring Completely

To factor a polynomial, **first factor the greatest common factor**, then consider the number of terms in the polynomial.

I. Two terms: Determine if the binomial is one of the following:

Difference of two squares: $a^2 - b^2 = (a + b)(a - b)$

Sum of two cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Difference of two cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

II. Three Terms: Determine if the trinomial is a perfect square trinomial.

a) If the trinomial is a perfect square, then

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

b) If the trinomial $ax^2 + bx + c$ is not a perfect square, then

i) if the leading coefficient $a = 1$, then check the product = c and sum = b to determine the correct combination.

ii) if the leading coefficient $a > 1$, then use trial and error or the ac-method.

III. Four terms: Try to factor by grouping.

Exercise 12: Factor each polynomial completely. This may mean factoring GCF and/or factoring in two or more steps.

a) Factor completely $3x^4 - 3x^3 - 36x^2$

b) Factor completely $20a - 5a^3$

c) Factor completely $16a^5b - ab$

d) Factor completely $8x^2 - 24x + 18$

e) Factor completely $16x^3y - 40x^2y^2 + 25xy^3$

f) Factor completely $12x^3 + 11x^2 + 2x$

g) Factor completely $7z^2w^2 - 10zw^2 - 8w^2$

Section 1.2: Radicals and the Complex Numbers

Simplifying Radicals

Exercise 1: Simplify each radical below

a. $\sqrt{9}$

b. $\sqrt{18}$

c. $\sqrt{25}$

d. $\sqrt{75}$

e. $\sqrt{49}$

f. $\sqrt{98}$

Exercise 2: Simplify each radical below

a. $\sqrt{x^2}$

b. $\sqrt{x^3}$

c. $\sqrt{x^4}$

d. $\sqrt{x^5}$

e. $\sqrt{x^6}$

f. $\sqrt{x^7}$

g. $\sqrt{x^{98}}$

h. $\sqrt{x^{99}}$

Write a rule for simplifying a radical of the form $\sqrt{x^n}$

When n is even, $\sqrt{x^n} =$

When n is odd, $\sqrt{x^n} =$

Exercise 3: Simplify each radical

a) $\sqrt{12x^6y^3}$

b) $3x\sqrt{20x^3y^6z^9}$

Helpful Radical Multiplication Rule:

Square of a Radical: $(\sqrt{a})^2 = (\sqrt{a})(\sqrt{a}) = a$

Product of radical conjugates: $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$

Strategy for rationalizing denominators with radicals:

If the denominator has one term, multiply the numerator and the denominator by the radical:

Example $\frac{2}{\sqrt{a}} = \frac{2}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{2\sqrt{a}}{a}$

If the denominator has two terms, multiply the numerator and the denominator by the conjugate of the denominator:

Examples a) $\frac{3}{\sqrt{a} + \sqrt{b}} = \frac{3}{\sqrt{a} + \sqrt{b}} \cdot \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{3(\sqrt{a} - \sqrt{b})}{a - b}$

Exercise 4: Rationalize denominators. Simplify the answer.

a) $\frac{9}{\sqrt{7}}$

b) $\frac{3}{2\sqrt{x}}$

c) $\frac{-12}{\sqrt{10}-\sqrt{6}}$

d) $\frac{\sqrt{6}}{5-\sqrt{2}}$

Complex numbers

We define the imaginary unit $i = \sqrt{-1}$ as the solution to $x^2 = -1$.

The square root of a negative number can be simplified $\sqrt{-b} = \sqrt{-1}\sqrt{b} = i\sqrt{b}$ where $b > 0$

Exercise 5: Simplify the expressions.

a. $\sqrt{-81}$

b. $\sqrt{-75}$

c. $-\sqrt{-49}$

d. $\sqrt{-15}$

Powers of i :

$i^1 = i$

$i^5 = \underline{\hspace{2cm}}$

$i^9 = \underline{\hspace{2cm}}$

$i^2 = -1$

$i^6 = \underline{\hspace{2cm}}$

$i^{10} = \underline{\hspace{2cm}}$

$i^3 = -i$

$i^7 = \underline{\hspace{2cm}}$

$i^{11} = \underline{\hspace{2cm}}$

$i^4 = 1$

$i^8 = \underline{\hspace{2cm}}$

$i^{12} = \underline{\hspace{2cm}}$

What is the pattern for powers of i ? Write a rule for it _____

Exercise 6: Simplify the product or quotient in terms of i

a) $(\sqrt{-9})(\sqrt{16})$

b) $\frac{\sqrt{-36}}{\sqrt{-9}}$

c) $(\sqrt{-12})(\sqrt{-6})$

d) $\frac{\sqrt{36}}{\sqrt{-2}}$

Caution:



Correct work: $(\sqrt{-12})(\sqrt{-6}) = (i\sqrt{12})(i\sqrt{6}) = i^2\sqrt{72} = -6\sqrt{2}$

Incorrect work: $(\sqrt{-12})(\sqrt{-6}) = \sqrt{72} = 6\sqrt{2}$

Correct work: $\frac{\sqrt{36}}{\sqrt{-2}} = \frac{\sqrt{36}}{i\sqrt{2}} = \frac{\sqrt{36}}{i\sqrt{2}} \cdot \frac{i}{i} = \frac{i\sqrt{36}}{i^2\sqrt{2}} = \frac{i\sqrt{18}}{-1} = -3i\sqrt{2}$

Incorrect work: $\frac{\sqrt{36}}{\sqrt{-2}} = \sqrt{-18} = i\sqrt{18} = 3i\sqrt{2}$

A complex number is a number of the form $x + yi$ where x and y are real numbers. We say x is the **real part** of the complex number and yi is the **imaginary part** of the complex number.

We can add, subtract, multiply, and divide complex numbers following similar procedures and the operations of radicals if the radicands are positive. If the radicands are negative, we must first convert them to i .

Exercise 7: Perform the indicated operation.

a) $(-5 + 9i) - (-2 + 3i)$

b) $(2 + 7i)(5 - 3i)$

c) $(2 + 3i)(2 - 3i)$

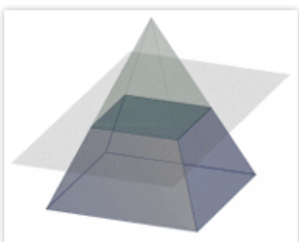
The complex numbers $a + bi$ and $a - bi$ are called **conjugates**.

The product of complex Conjugates $(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$ is a real number.

Exercise 8: Rationalize the denominator.

a) $\frac{5+i}{2i}$

b) $\frac{3i}{5-2i}$



Complex number or imaginary number concept was first investigated by a mathematician and inventor named Heron (c. 10-70 A.D.) from the city of Alexandria on the coast of the Mediterranean, in Egypt. While trying to find the volume of the frustum of a pyramid (see Figure 1) with a square base of a certain size, Heron of Alexandria first encountered the square root of a negative number (Nahin, 1998).

Section 1.3 Methods for Solving Quadratic Equations

A quadratic function is of the form $f(x) = ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$. The roots of the function are solutions to the equation $ax^2 + bx + c = 0$.

The quadratic equation $ax^2 + bx + c = 0$ can be solved in many ways:

1. By factoring and the Zero Factor Theorem
2. By the Square root property
3. By the Quadratic Formula

Method 1: Solving Quadratic Equations by Factoring

Zero Factor Theorem: If $ab = 0$ then $a = 0$ or $b = 0$ or both.

Solving Quadratic Equations by Factoring (Use this method when the equation is factorable)

1. Put the equation in standard form: $ax^2 + bx + c = 0$.
2. Factor completely.
3. Use the zero-product rule, set each factor containing the variable equal to zero and solve for x . **Note: Do not solve for the constant factor.**

Exercise 1: Solve the equations by factoring

a) Solve: $x^2 - 4x - 32 = 0$

b) Solve: $36 - 49y^2 = 0$

c) Solve: $a^3 - 6a^2 + 5a = 0$

d) Solve: $2x^3 + 9x^2 = 5x$

Method 2: Solving Quadratic Equations by the Square Root Property

The Square Root Property

$$\text{If } x^2 = a, \text{ then } x = \sqrt{a} \text{ or } x = -\sqrt{a}.$$

Sometimes it is indicated as $x = \pm\sqrt{a}$

Solving Quadratic Equations by the Square Root Property (Use this method when the equation consists of a square term and a constant term)

- Isolate the square and then apply the Square Root Property

Exercise 1: Solve using the square root Property. Simplify the radical. Write out both solutions.

a) $x^2 = 100$

b) $s^2 - 12 = 0$

c) $s^2 + 27 = 0$

d) $(y - 3)^2 - 98 = 0$

iv) $2(x + 1)^2 + 48 = 0$

Method 3: Solving Quadratic Equations by the Quadratic Formula

The Quadratic Formula

The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solving Quadratic Equations by the Quadratic Formula (This method can be used on any quadratic equations; it may not be the best method if the equation can be solved using the other two methods.)

1. Put the equation in standard form: $ax^2 + bx + c = 0$.
2. Substitute, a , b , and c into the quadratic formula and simplify.

Exercise 1: Solve using the quadratic formula

a) $x^2 - 5x - 8 = 0$

b) $x^2 + 2x + 6 = 0$

c) $2x^2 + 5 = 4x$

The **discriminant** is the radicand $b^2 - 4ac$ in the quadratic formula.

If the discriminant > 0 , then the equation has two real roots.

If the discriminant $= 0$, then the equation has one real (double) root.

If the discriminant < 0 , then the equation has two complex roots.

Exercise 2: Solve $4x^2 - 28x + 49 = 0$. What is the best method for solving this equation? Factoring, the square root property, or the quadratic formula? What type of roots does it have?

Exercise 3: Solve $x^2 + 63 = 0$. What is the best method for solving this equation? Factoring, the square root property, or the quadratic formula? What type of roots does it have?

Exercise 4: Solve $2x^2 - 6x - 9 = 0$. What is the best method for solving this equation? Factoring, the square root property, or the quadratic formula? What type of roots does it have?

SECTION 1.1 SUPPLEMENTARY EXERCISES

1. Factor completely

- | | | |
|---------------------|----------------------------|-----------------------------|
| a) $a^2 - 5a$ | h) $49 - 9t^2$ | o) $2x^3y + x^2y^2 - 6xy^3$ |
| b) $25y^3 - 15y^2$ | i) $36x^2y - 9y$ | p) $-6x^2 - 9x + 15$ |
| c) $5ab^2 - 15a^3b$ | j) $a^2 - 4a - 12$ | q) $am - 5a + 2bm - 10b$ |
| d) $64w^2 - 81$ | k) $x^2 + 9xy - 36y^2$ | r) $15x - 12ax + 10y - 8ay$ |
| e) $36x^2 - 25y^2$ | l) $m^3 - 4m^2 - 21m$ | s) $7b - 2bd + 21c - 6cd$ |
| f) $16x - 4x^3$ | m) $24 + 5n - n^2$ | t) $16a^5 - 250a^2$ |
| g) $y^3 - 81y$ | n) $2x^3 - 2x^2y - 12xy^2$ | |

SECTION 1.3 SUPPLEMENTARY EXERCISES

1. Solve the equations by factoring

- | | | |
|-------------------------|-----------------------------|-------------------------------|
| a) $x^2 + 6x + 8 = 0$ | f) $2y^3 + 12y^2 - 32y = 0$ | k) $5m^2 + 20m = 6 - 9m$ |
| b) $2t^2 + 3t - 14 = 0$ | g) $k^2 - 25 = 0$ | l) $(y + 5)^2 - 4 = 0$ |
| c) $x^2 - 10x + 25 = 0$ | h) $16x^2 = 49$ | m) $(n - 3)(3n - 2) - 8n = 0$ |
| d) $w^2 + 14w + 55 = 6$ | i) $z^2 - 9z = 0$ | n) $10x^2 = 27x - 18$ |
| e) $t^2 - 5t = 24$ | j) $-6x^2 - x + 12 = 0$ | |

2. Solve the equations by the square root property

- | | |
|------------------------------------|----------------------------|
| a) $x^2 = 120$ | f) $2k^2 + 80 = 0$ |
| b) $s^2 - 45 = 0$ | g) $(y - 3)^2 + 64 = 0$ |
| c) $(y + 2)^2 - 49 = 0$ | h) $y^2 + 88 = 7$ |
| d) $\frac{1}{2}(x - 6)^2 - 16 = 0$ | i) $(y + 7)^2 + 50 = 0$ |
| e) $8 = (b - 5)^2 - 16$ | j) $(y - 1)^2 + 12 = 0$ |
| | k) $(2k - 1)^2 + 72 = 18$ |
| | l) $9(2m - 3)^2 + 8 = 449$ |

3. Solve the equations by the quadratic formula

- | | |
|--------------------------|-----------------------------------|
| a) $p^2 + 5p = -2$ | h) $4x^2 - x + 6 = 0$ |
| b) $a^2 - 2a = 4$ | i) $m^2 + 2m = -5$ |
| c) $2k^2 = -4k + 3$ | j) $2k^2 + 9k = -7$ |
| d) $12m + 9m^2 = -4$ | k) $(z - 2)(z + 4) + 6 = 0$ |
| e) $3x^2 + 6x + 2 = 0$ | l) $(3x - 7)(x + 5) = -31$ |
| f) $6x^2 + 2x + 3 = 0$ | m) $11n^2 - 4n(n - 2) = 6(n + 3)$ |
| g) $10x^2 - 13x - 3 = 0$ | n) $x(x - 3) = -10x - 7$ |

Module II

Polynomial Functions

Section 2.1 Definition of Functions, Domain, Range, and The Interval Notation

A function f is an assignment (or relation) that assigns to each input x in the domain exactly one output y in the codomain.

The domain of a function f is the set of all possible inputs of f . The codomain of a function is the set of all possible outputs of f . The range of a function f is a subset of codomain and the actual outputs of f .

For example,

$f: \mathbb{R} \rightarrow \mathbb{R}$ means the function f has a **domain** of \mathbb{R} (all real numbers) and the **codomain** \mathbb{R} (all real numbers).

If we define f as the assignment $f: x \mapsto x^2$, then f takes an input x and assigns it to the output x^2 . We can also write it as

$$f(x) = x^2 \quad \text{or} \quad y = x^2$$

In this case, the **range** of f is the set of nonnegative real numbers, or $[0, \infty)$ using the interval notation.

A function may assign the same output to two different inputs, but it may not assign two outputs to one input. In other words, two different inputs may have the same output, but one input may not have two different outputs.

Take the vending machine for example, one code (an input) returns one bottle of beverage. It won't return two bottles of beverages unless the machine malfunctions. On the other hand, it is possible that two different codes (different inputs) give us the same beverage (output).



Exercise 1. Determine if each of the following set forms a function. If it is a function, determine its domain.

	Function? (Yes or No)	Why or Why Not	If a function, what is its domain?
Set of ordered pairs $\{ (1, 1), (2, 1), (3, 1) \}$			
Set of ordered pairs $\{ (1, 1), (1, 2), (1, 3) \}$			
Fahrenheit-Celsius conversion formula $F = \frac{9}{5}C + 32$			
Equation of a line: $y = mx + b$			
Parabola: $y = ax^2 + bx + c$			
Circle: $x^2 + y^2 = 1$			

Interval on the real number line

Inequality notation	Real number line	Interval notation
$a \leq x \leq b$		$[a, b]$
$a < x < b$		(a, b)
$a \leq x < b$		$[a, b)$
$a < x \leq b$		$(a, b]$
$x \geq a$		$[a, \infty)$
$x > a$		(a, ∞)
$x \leq b$		$(-\infty, b]$
$x < b$		$(-\infty, b)$
All real numbers		$(-\infty, \infty)$

Section 2.2 Quadratic Functions: Graphs and Roots

A **quadratic function** is of the form $f(x) = ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$.

The graph of a quadratic function is a parabola (U-shaped). A parabola has either a maximum or a minimum point, called the **vertex**. The **axis of symmetry** for the graph of a quadratic function is the vertical line through the vertex.

The vertex can be found by the following methods:

Method 1: The Vertex Form

A quadratic function expressed as $f(x) = a(x - h)^2 + k$ is said to be in the **vertex form**, where (h, k) is the vertex.

Method 2: The Vertex Formula

If a quadratic function is in standard form $f(x) = ax^2 + bx + c$, the vertex can be computed using the formula:

$$\text{x-coordinate of the vertex is } -\frac{b}{2a}$$

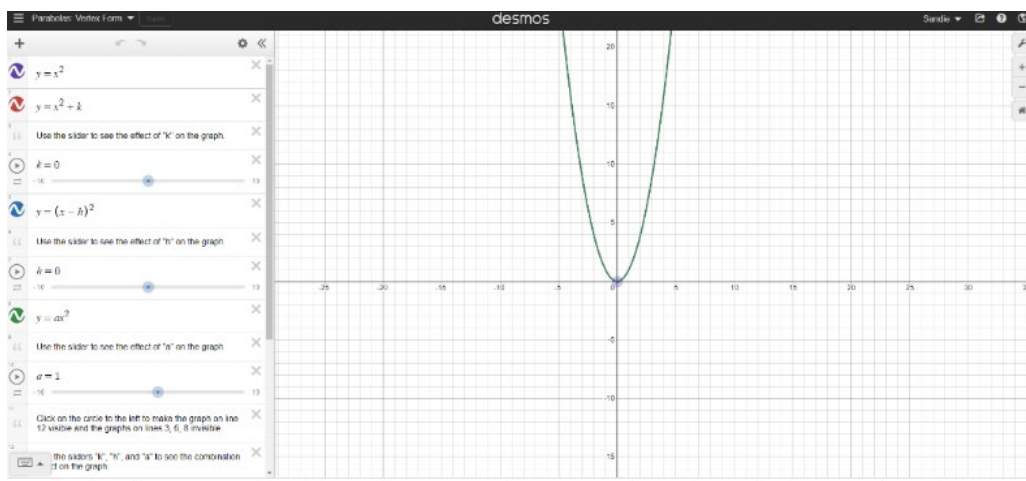
Substitute x-coordinate to find the y-coordinate of the vertex

The **roots** of a quadratic function are those x values such that $f(x) = 0$, or solutions to the equation $ax^2 + bx + c = 0$. They are also called the **zeros** of the function. The quadratic equation $ax^2 + bx + c = 0$ can be solved by factoring, by the square root property, or by the quadratic formula. See section 1.3 for methods of solving quadratic equations.



Use the Desmos link below to see the effect of changing h , k and a on the parabola.

<https://www.desmos.com/calculator/0txid19ts5>



How does the graph change when you change h ? When h is positive? When h is negative?

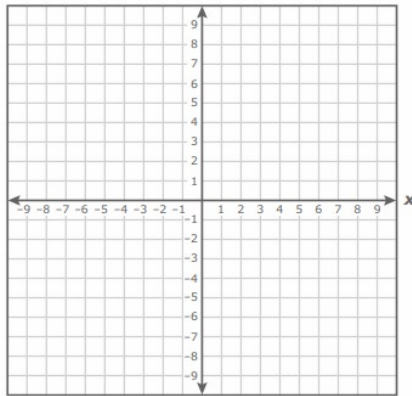
How does the graph change when you change k ? When k is positive? When k is negative?

How does the graph change when you change a ? When a is positive? When a is negative? When $|a| > 1$? When $|a| < 1$?

Exercise 1: For each quadratic function:

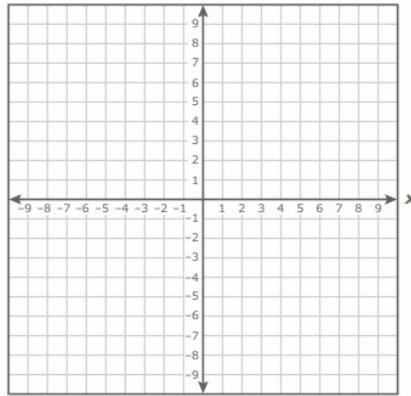
- i) Identify the vertex.
- ii) Sketch the graph.
- iii) Determine the domain and the range of the function.
- iv) Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.

a) $f(x) = x^2$



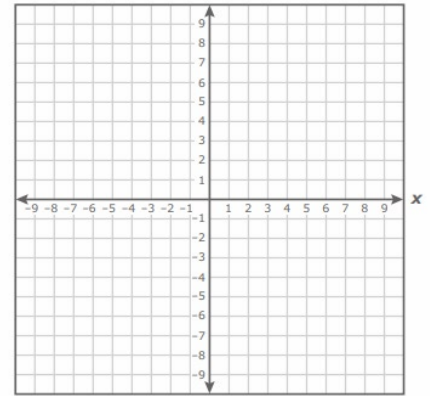
The vertex is _____
 The Domain is _____
 The Range is _____
 The roots are _____

b) $f(x) = x^2 - 3$



The vertex is _____
 The Domain is _____
 The Range is _____
 The roots are _____

c) $f(x) = (x - 3)^2$

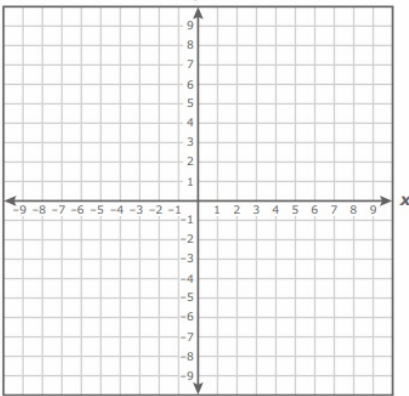


The vertex is _____
 The Domain is _____
 The Range is _____
 The roots are _____

Exercise 2: For each quadratic function,

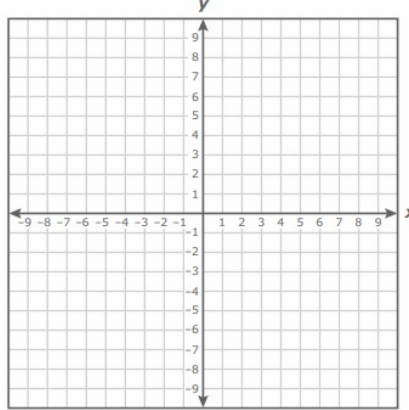
- Use the **Vertex Form** to determine the vertex.
- Sketch the graph.
- Determine the domain and the range of the function.
- Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.

a) $f(x) = -(x + 5)^2 + 6$



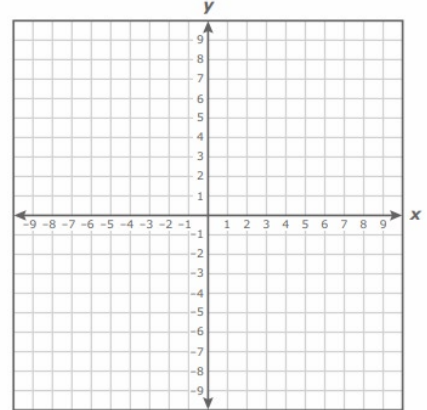
The vertex is _____
 The Domain is _____
 The Range is _____
 The roots are _____

b) $f(x) = 2(x + 2)^2 - 8$



The vertex is _____
 The Domain is _____
 The Range is _____
 The roots are _____

c) $f(x) = -\frac{1}{2}x^2 - 2$

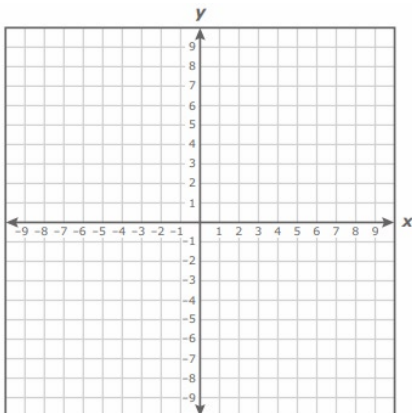


The vertex is _____
 The Domain is _____
 The Range is _____
 The roots are _____

Exercise 3: For each quadratic function,

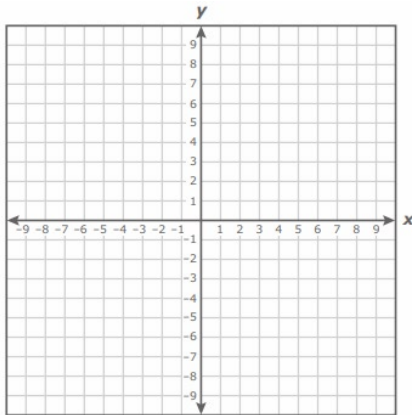
- Use the **Vertex Formula** to determine the vertex.
- Sketch the graph.
- Determine the domain and the range of the function.
- Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.
- Determine the interval where the function is positive or negative

a) $f(x) = x^2 + 4x - 5$



The vertex is _____
 The Domain is _____
 The Range is _____
 The roots are _____
 The function is positive on the interval _____
 The function is negative on the interval _____

b) $f(x) = -x^2 - 2x - 5$



The vertex is _____

The Domain is _____

The Range is _____

The roots are _____

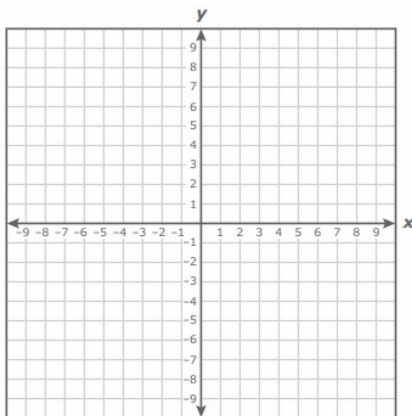
The function is positive on the interval _____

(use decimal approximation)

The function is negative on the interval _____

(use decimal approximation)

c) $f(x) = x^2 - 2x + 5$



The vertex is _____

The Domain is _____

The Range is _____

The roots are _____

The function is positive on the interval _____

The function is negative on the interval _____

How do you identify the real roots of the function? _____

What do you notice about the roots in exercises 2f and 3c? How are their graphs different from the rest? What can you say about the graph of a quadratic function with complex roots?

Section 2.3 Polynomial Functions: Graphs and Roots

A polynomial function is a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers. a_n is the leading coefficient, and a_0 is the constant term, and n is the degree of the polynomial.

The **roots** of a polynomial function are those x values such that $f(x) = 0$. They are also called the **zeros** or **solutions** of the polynomial function.

A polynomial function has the following characteristics:

- A polynomial function with degree n has n roots (including multiplicity), they may be real or complex roots.
- A polynomial function may cross the x -axis at most n times, corresponding to at most n distinct real roots.
- If a polynomial function with real coefficients has complex roots, the complex roots always exist in conjugate pair.
- A polynomial function may have a graph with at most $n - 1$ turning points.

The END behavior

The END behavior of the graph of a polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

is determined by n and a_n , where n is the degree of the polynomial and a_n is the leading coefficient of the polynomial.

Degree of the polynomial	Sign of the Leading Coefficient	Left End Behavior As $x \rightarrow -\infty$	Right End Behavior As $x \rightarrow \infty$	A maximum or minimum exists?	Domain & Range	Example
n is odd	a_n is positive	$f(x) \rightarrow -\infty$ or Down	$f(x) \rightarrow \infty$ or Up	No	$D = (-\infty, \infty)$ $R = (-\infty, \infty)$	
n is odd	a_n is negative	$f(x) \rightarrow \infty$ or Up	$f(x) \rightarrow -\infty$ or Down	No	$D = (-\infty, \infty)$ $R = (-\infty, \infty)$	
n is even	a_n is positive	$f(x) \rightarrow \infty$ or Up	$f(x) \rightarrow \infty$ or Up	There is a minimum, m	$D = (-\infty, \infty)$ $R = [\mathbf{m}, \infty)$	
n is even	a_n is negative	$f(x) \rightarrow -\infty$ or Down	$f(x) \rightarrow -\infty$ or Down	There is a maximum, M	$D = (-\infty, \infty)$ $R = (-\infty, \mathbf{M}]$	

Synthetic Division

The method of Synthetic Division is a quick way to divide a polynomial $f(x)$ by a linear factor of the form $(x - k)$. Instead of using long division, we can simply work with the coefficients of $f(x)$ and k .

Example: If $f(x) = ax^2 + bx + c$. Divide $f(x)$ by $(x - k)$.

Step 1: Set up synthetic division using the coefficients a, b, c and k .

$$\begin{array}{r|rrr} k & a & b & c \\ \hline \end{array}$$

Step 2: Bring down the first coefficient a . Multiply a by k and add to b

$$\begin{array}{r|rrr} k & a & b & c \\ & & ka & \\ \hline & a & b + ka & \end{array}$$

Step 3: Repeat the procedure. Bring down $b + ka$. Multiply $b + ka$ by k , and add to c .

$$\begin{array}{r|rrr} k & a & b & c \\ & & ka & k(b + ka) \\ \hline & a & b + ka & c + k(b + ka) \end{array}$$

Step 4: Repeat the procedure until we finish the last coefficient. The result of the division is a quotient polynomial $q(x) = ax + (b + ka)$ with remainder $c + k(b + ka)$. Note: The degree of the quotient is 1 less than the degree of the original polynomial.

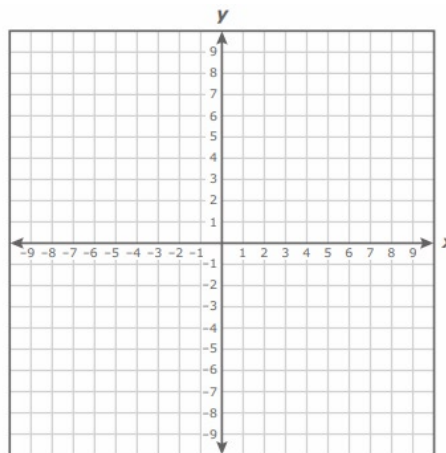
$$\begin{array}{r|rrr} k & a & b & c \\ & & ka & k(b + ka) \\ \hline & a & b + ka & c + k(b + ka) \end{array}$$

Coefficients of $q(x)$ \rightarrow a $b + ka$ $c + k(b + ka)$ \leftarrow Remainder

Exercise 1. If $f(x) = 3x^2 - 8x + 17$ is divided by $x - 4$. Determine the quotient and the remainder.

Exercise 2. If $f(x) = x^3 - 2x^2 - 5x + 6$.

- Divide $f(x)$ by $x + 2$ using synthetic division. Determine the quotient and the remainder. What is the significance of the remainder being zero?
- Set the quotient to zero and solve.
- Express f in factored form.
- Find all three roots of f . What is the characteristic of the roots? (Rational, irrational, complex?)
- Sketch a graph of f , check the graph on Desmos or graphing calculator. How do the roots of the function relate to its graph?



The Remainder and Factor Theorems

The Remainder Theorem:

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $f(k)$.

The Factor Theorem:

If a polynomial $f(x)$ has a factor $x - k$, if and only if $f(k) = 0$.

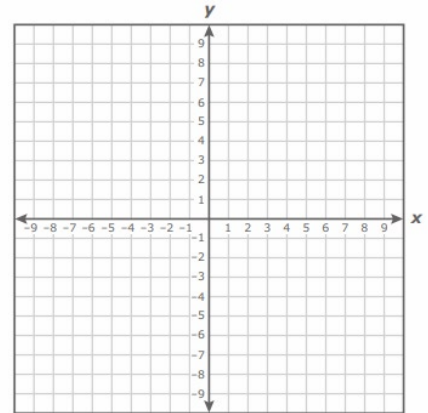
(i.e., if the remainder is zero, then the polynomial is factorable.)

The following are equivalent:

- r is a root of $f(x)$
- r is a zero of $f(x)$
- r is a solution of $f(x)$
- $x - r$ is a factor of $f(x)$
- $f(r) = 0$
- The remainder of $f(x)$ divided by $x - k$ is zero

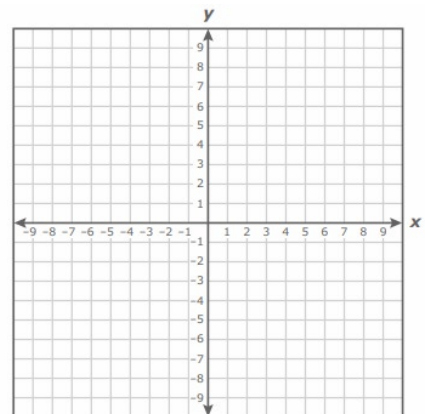
Exercise 3. If $f(x) = x^3 + 2x^2 + 4x + 8$

- Find a rational root r of the polynomial function using Desmos or graphing calculator.
- Use synthetic division to divide $f(x)$ by $x - r$.
- Find the remaining two roots by solving the quadratic function. Express f in factored form. What is the characteristic of the roots? (Multiplicity?)
- Sketch a graph. How do the roots of the function relate to its graph?



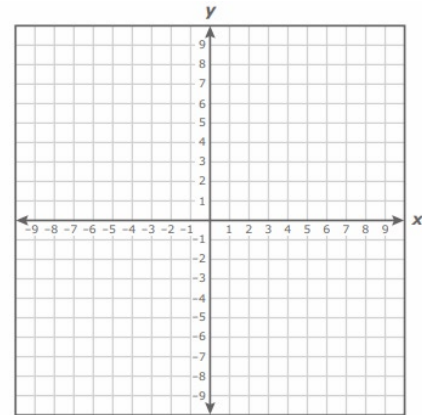
Exercise 4. If $f(x) = x^3 - x^2 - 10x + 12$

- Find a rational root r of the polynomial function using Desmos or graphing calculator.
- Use synthetic division to divide $f(x)$ by $x - r$.
- Find the remaining two roots by solving the quadratic function. What is the characteristic of the roots? (Rational, irrational, complex?)
- Sketch a graph. How do the roots of the function relate to its graph?



Exercise 5. If $f(x) = x^4 + x^3 - x^2 + x - 2$

- Find two rational roots r_1 and r_2 of the polynomial function using Desmos or graphing calculator.
- Apply synthetic division twice: First divide $f(x)$ by $x - r_1$, then divide the resulting quotient by $x - r_2$.
- Find the remaining two roots by solving the quadratic function. What is the characteristic of the roots? (Rational, irrational, complex?)
- Sketch a graph. How do the roots of the function relate to its graph?



Exercise 6. Find the polynomial of degree 3 with roots 1, 2, -3, and $f(-1) = 8$.

- Express the polynomial in factored form $f(x) = a(x - r_1)(x - r_2)(x - r_3)$.
- Use $f(-1) = 8$ to solve for a .

Exercise 7. Find a polynomial of degree 4 with real coefficients and roots $2i$ and 3. Root 3 has multiplicity 2.

SECTION 2.2 SUPPLEMENTARY EXERCISES

For each quadratic function,

- i. Determine the vertex.
- ii. Sketch the graph.
- iii. Determine the domain and the range of the function.
- iv. Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.

a) $f(x) = (x - 3)^2 - 1$

b) $f(x) = -(x - 1)^2 + 7$

c) $f(x) = \frac{1}{3}(x - 8)^2 - 7$

d) $f(x) = -(x - 2)^2 + 9$

e) $f(x) = 6(x + 5)^2 + 4$

f) $f(x) = \frac{1}{2}(x + 5)^2 - 3$

g) $f(x) = -(x - 3)^2 - 1$

h) $f(x) = 4(x + 7)^2 + 8$

i) $f(x) = 2x^2 + 4x - 5$

j) $f(x) = -x^2 + 4x - 6$

SECTION 2.3 SUPPLEMENTARY EXERCISES

1. a) Divide $x^3 + 2x^2 - 6x - 9$ by $x - 2$

b) Divide $x^3 + 2x^2 - 6x - 9$ by $x + 3$

c) Divide $2x^3 - 3x^2 - 5x + 18$ by $x + 2$

2. Find the zeros of a function

a) One zero of $f(x) = x^3 - 2x^2 - 9x + 18$ is $x = 2$. Find the other zeros.

b) One zero of $f(x) = 4x^3 + 9x^2 - 52x + 15$ is $x = -5$. Find the other zeros.

c) Find all the real zeros for $f(x) = x^4 + 4x^3 - 9x^2 - 36x$.

d) Find all the real zeros for $f(x) = x^4 - 13x^2 + 36$.

3. Find a polynomial of degree 4 with real coefficients and roots i and $2 + i$.

Module III

Exponential and Logarithmic Functions

Section 3.1 Exponential Functions and Graphs

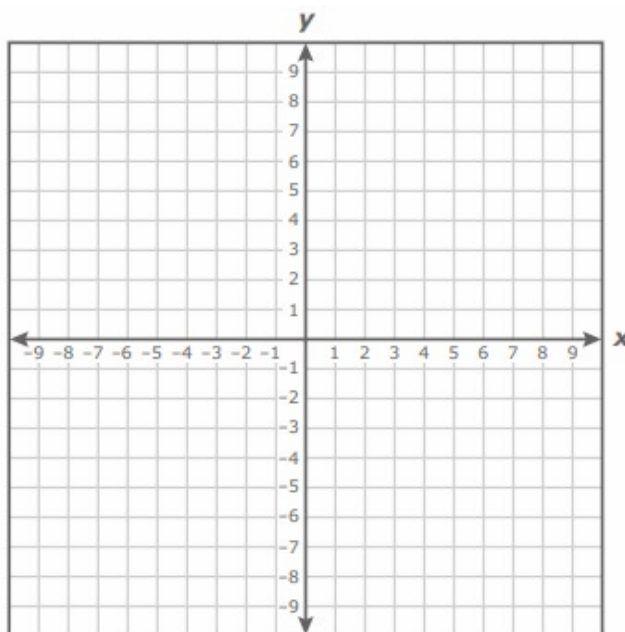
For any real number x , an exponential function is a function of the form

$$f(x) = c \cdot b^x$$

Where the base b is a positive real number such that $b \neq 1$ and c is a non-zero real number.

Example 1: Graph the exponential function: $f(x) = 2^x$. What is its domain and range?

x	$y = f(x)$
-3	
-2	
-1	
0	
1	
2	
3	



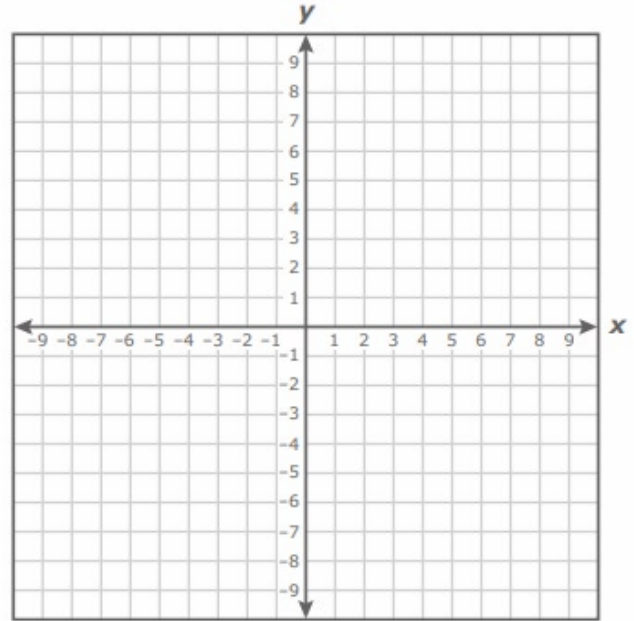
Characteristics of the exponential graph: $y = c \cdot b^x$

- The exponential graph is always increasing if $b > 1$; it is always decreasing if $0 < b < 1$.
- The graph does not cross the x -axis; $y = 0$ is a horizontal asymptote.
- The graph has the y -intercept $(0,1)$.
- The domain of the exponential function is all real numbers $(-\infty, \infty)$.
- If $c > 0$, the range of the exponential function is $(0, \infty)$. If $c < 0$, the range is $(-\infty, 0)$.

Example 2:

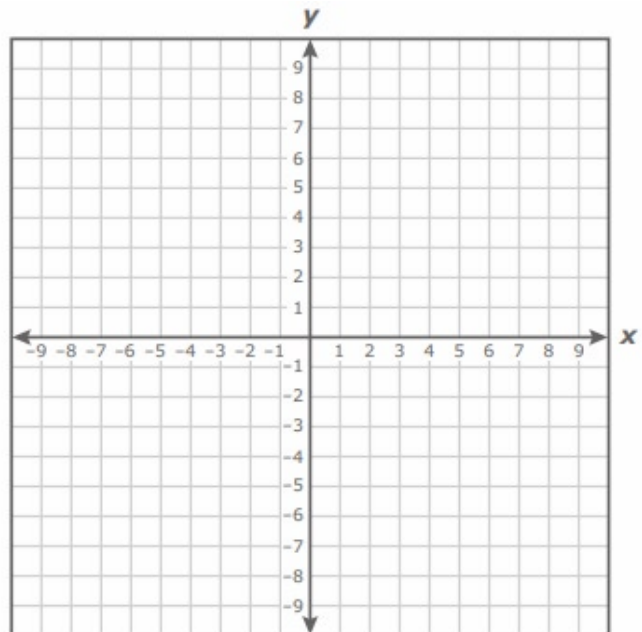
- a) Graph the exponential function: $g(x) = \left(\frac{1}{2}\right)^x$. What is its domain and range?
b) Graph the exponential function: $h(x) = 2^{-x}$. What is its domain and range?
c) How do the two graphs compare? Why is it so?

x	$g(x) = \left(\frac{1}{2}\right)^x$	$h(x) = 2^{-x}$
-3		
-2		
-1		
0		
1		
2		
3		



Example 3: Graph the exponential function: $k(x) = -2^x$. What is its domain and range?

x	$y = k(x)$
-3	
-2	
-1	
0	
1	
2	
3	



Section 3.2 Logarithmic Functions and Graphs

If $b > 0$ and $b \neq 1$, then $y = \log_b x \Leftrightarrow x = b^y$

Exercise 1: Evaluate the logarithms and rewrite each logarithmic relation as an exponential relation.

Evaluate the logarithm	Rewrite in exponential form
$\log_3 9 = ?$	$\log_3 9 = 2 \Leftrightarrow 3^2 = 9$
$\log_3 1$	
$\log_3 \frac{1}{9} =$	
$\log_2 8 =$	
$\log_8 2 =$	
$\log_{\frac{1}{2}} 2 =$	

Exercise 2: Evaluate the logarithms.

a) $\log_{10} 1$

b) $\log_{10} 10$

c) $\log_{10} 100$

d) $\log_{10} 1000$

e) $\log_{10} \left(\frac{1}{10}\right)$

f) $\log_{10} \left(\frac{1}{100}\right)$

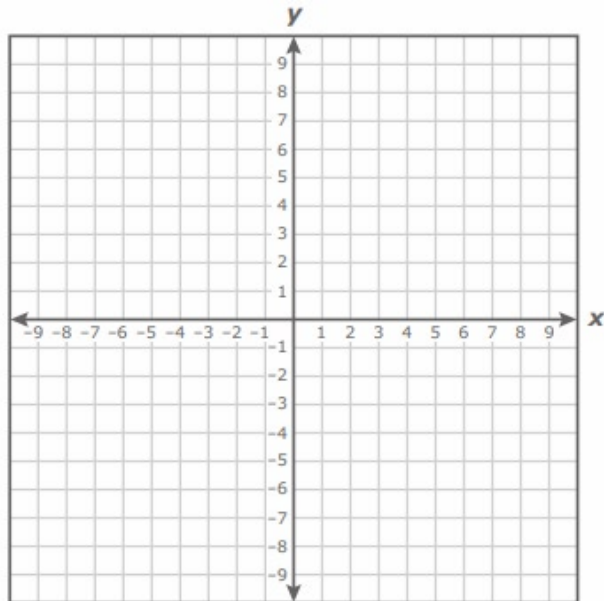
g) $\log_{10} \left(\frac{1}{1000}\right)$

Exercise 3: Graph the logarithmic function: $y = \log_2 x$.

Convert logarithm to exponential form; complete the table of values and graph. What is its domain and range?

$$y = \log_2 x \Leftrightarrow x = 2^y$$

x	y
	-3
	-2
	-1
	0
	1
	2
	3

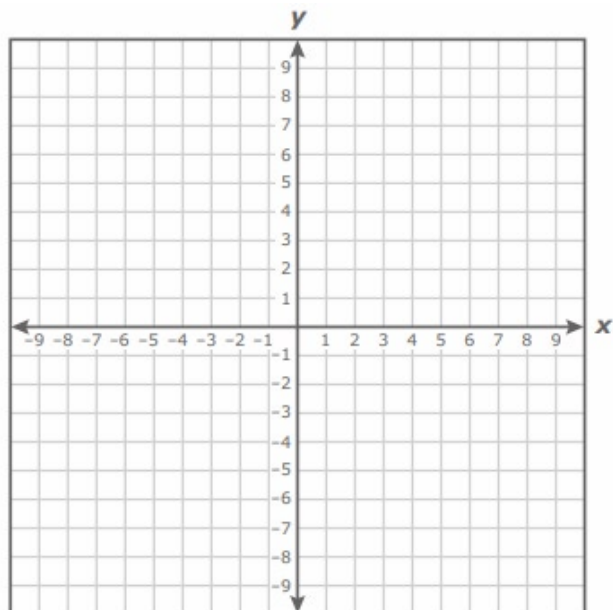


Exercise 4: Graph the logarithmic function: $y = \log_3 x$.

Convert logarithm to exponential form; complete the table of values and graph. What is its domain and range?

$$y = \log_3 x \Leftrightarrow x = 3^y$$

x	y
	-2
	-1
	0
	1
	2



Characteristics of the logarithmic function: $f(x) = \log_b x$

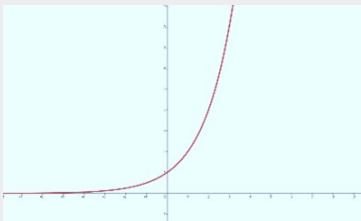
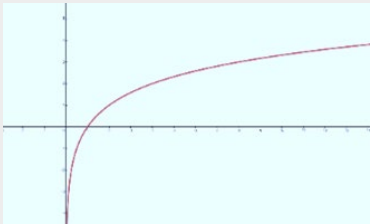
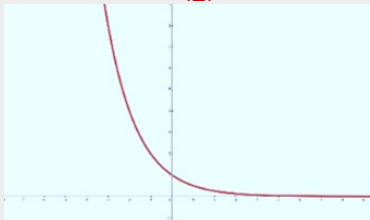
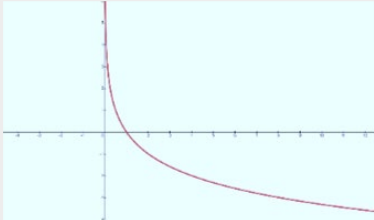
- The function does not cross the y -axis; $x = 0$ is a vertical asymptote.
- The function has the x -intercept $(1, 0)$.
- The domain of the logarithmic function is $(0, \infty)$.
- The range of the logarithmic function is $(-\infty, \infty)$.



Use the link to try the Desmos activity: <https://www.desmos.com/calculator/bvh8odgu6n>

How do the exponential and logarithmic graphs change when the base is >1 ? When the base is between 0 and 1? _____

What is the inverse symmetry between the exponential and the logarithmic functions of the same base? _____

Comparing the Graphs of $y = b^x$ and $y = \log_b x$ ($b > 0, b \neq 1$)		
	$y = b^x$	$y = \log_b x$
Domain	$(-\infty, \infty)$	$(0, \infty)$
Range	$(0, \infty)$	$(-\infty, \infty)$
x -intercept	None	$(1, 0)$
y -intercept	$(0, 1)$	None
Contains points	$(0, 1), (1, b)$	$(1, 0), (b, 1)$
Asymptote	x -axis or horizontal line $y = 0$	y -axis or vertical line $x = 0$
Example $b > 1$ (Always increasing)	$y = 2^x$ 	$y = \log_2 x$ 
Example $0 < b < 1$ (Always decreasing)	$y = \left(\frac{1}{2}\right)^x$ 	$y = \log_{1/2} x$ 

Section 3.3 Properties of Logarithms

Properties of Logarithms

	Properties of Exponents	Corresponding Properties of Logarithms
Product Property	$b^{x+y} = b^x \cdot b^y$	$\log_b(x \cdot y) = \log_b x + \log_b y$
Quotient Property	$b^{x-y} = \frac{b^x}{b^y}$	$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
Power Property	$(b^x)^n = b^{nx}$	$\log_b(x^n) = n \cdot \log_b x$
	$b^1 = b$	$\log_b(b) = 1$
	$b^0 = 1$	$\log_b(1) = 0$

Exercise 1: Use the properties of logarithm and the values $\log_5 3 \approx 0.683$ and $\log_5 7 \approx 1.209$ to evaluate each of the following. (What is the value of $\log_5 5$?)

a) $\log_5 21$

b) $\log_5 \frac{3}{7}$

c) $\log_5 35$

d) $\log_5 \frac{15}{7}$

e) $\log_5 49$

The logarithm with base 10 is called the **common logarithm** denoted by \log_{10} or \log

$$\log_{10} x = \log x$$

The logarithm with base e is called the **natural logarithm** denoted by \log_e or \ln

$$\log_e x = \ln x$$

The Number e

The **natural base e** is a mathematical constant defined as the value $\left(1 + \frac{1}{n}\right)^n$ as n approaches ∞ . Its value is approximately

$$e \approx 2.71828182845$$

Properties of \log and \ln

log	ln
$\log(x \cdot y) = \log x + \log y$	$\ln(x \cdot y) = \ln x + \ln y$
$\log\left(\frac{x}{y}\right) = \log x - \log y$	$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
$\log x^n = n \log x$	$\ln x^n = n \ln x$
$\log 10 = 1$	$\ln e = 1$
$\log 1 = 0$	$\ln 1 = 0$

Exercise 2: Expand using the properties of logarithms. Assume x and y are positive.

a) $\log_3\left(\frac{9\sqrt{x}}{y^2}\right)$

b) $\log\frac{(x+3)}{100}$

c) $\ln\left(\frac{6e^4}{4}\right)$

Exercise 3: Condense into one single logarithm.

a) $\log 6 + 2 \log 2 - \log 3$

c) $9 \log a - 4 \log 2$

e) $7 \ln x + 6 \ln(x + 4) - 3 \ln y$

The inverse relation of a logarithmic and exponential functions:

$$\log_b b^x = x$$

$$b^{\log_b x} = x$$

The inverse relation in terms of the common log and the natural log:

$$\log 10^x = x$$

$$\ln e^x = x$$

$$10^{\log x} = x$$

$$e^{\ln x} = x$$

Exercise 4: Simplify the expressions. Assume x and y are positive.

a) $10^{\log 2^3}$

b) $\log_3 9^x$

c) $\ln e^{(3x+y)}$

d) $e^{2 \ln x}$

4.3 Solving Exponential and Logarithmic Equations

Let $b > 0$ and $b \neq 1$. Then

$$b^x = b^y \Leftrightarrow x = y.$$

$$\log_b x = \log_b y \Leftrightarrow x = y$$

Solving Exponential Equations

If it's possible to write both sides of the equation as a power of the **same** base, use the relation $b^x = b^y \Leftrightarrow x = y$ to set the exponents equal to each and solve.

Example 1: Solve $9^x = 27$

$$(3^2)^x = (3)^3$$

$$(3)^{2x} = (3)^3$$

$$\Rightarrow 2x = 3$$

$$\Rightarrow x = \frac{3}{2}$$

If it's **not** possible to write both sides of the equation as a power of the **same** base, use the relation $\log x = \log y \Leftrightarrow x = y$ and the power property $\log x^n = n \log x$ to apply **log** on both sides of the equation, simplify, and solve.

Example 2: Solve $9^x = 26$

$$\log 9^x = \log 26$$

$$x \cdot \log 9 = \log 26$$

$$\Rightarrow x = \frac{\log 26}{\log 9} \approx 1.4828$$

Repeat Example 2, this time, apply **ln** instead of **log**:

$$\ln 9^x = \ln 26$$

Use the calculator to find the decimal approximation. How does the answer compare with above?

Exercise 1: Solve by rewriting both sides of the equation as a power of the **same** base

a) Solve: $2^{x+1} = 8$

b) Solve: $16^{x+1} = 64$

c) Solve: $\left(\frac{1}{6}\right)^x = 36$

d) Solve: $4^{2x+3} = 8^{x+2}$

Exercise 2: Solve by applying either **log** or **ln** on both sides of the equation. Round answer to the nearest ten thousandths (four decimal places).

a) Solve $2^x = 5$

b) Solve $41^x = 25$

Solving Logarithmic Equations

First, use properties of logarithms to combine log terms or simplify to one of the following forms:

1. If **log** appears only on one side of the equation in the form of $\log_b x = y$, rewrite it as an exponential equation $b^y = x$ and solve.
2. If **log** appears on both sides of the equation in the form of $\log_b x = \log_b y$, set $x = y$ and solve.

Exercise 3: Solve each logarithmic equation

a) Solve: $\log_6 x = 2$

b) Solve: $\log_x 32 = 5$

c) Solve: $\log_7 23 = x$

d) Solve: $\log_3(5x - 1) = \log_3(x + 7)$

e) Solve: $\log x + \log(x - 15) = 2$ **Remember to check for extraneous solutions!**

SECTION 3.3 SUPPLEMENTARY EXERCISES

1. Expand using the properties of logarithms. Assume x and y are positive.

a) $\log_2 \frac{7x^3}{y}$

b) $\log_5 \frac{5a^8}{b^3}$

2. Condense into a single logarithm.

a) $5 \log x + 2 \log y - 4 \log z$

b) $\frac{1}{3} \log w + \frac{1}{2} \log p - 3 \log 5$

SECTION 3.4 SUPPLEMENTARY EXERCISES

1. Solve for x

a) Solve: $4^x = 32$

b) Solve: $27^{x-2} = 9^{x+1}$

c) Solve: $3^x = 67$

d) Solve: $58^x = 38$

2. Evaluate and round answer to the nearest ten thousandths (four decimal places).

i. Set the expression equal to x . Rewriting it as an exponential equation and solve.

ii. Or use **Change of Base Formula**: $\log_b x = \frac{\log x}{\log b}$ or $\log_b x = \frac{\ln x}{\ln b}$

a) $\log_3 7$

b) $\log_2 6$

c) $\log_8 5$

3. Solve the logarithmic equation. **Remember to check for extraneous solutions!**

a) Solve: $\log_5(3x + 1) = 2$

b) Solve: $\log_x \left(\frac{1}{64} \right) = 3$

c) Solve: $\log_6 x + \log_6(x + 1) = 2$

d) Solve: $\log 5x + \log(x - 1) = 2$

e) Solve: $2 - \log(x - 1) = \log(x - 16)$

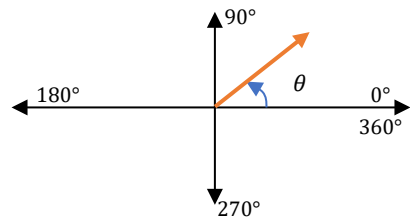
Module IV

Trigonometric Functions

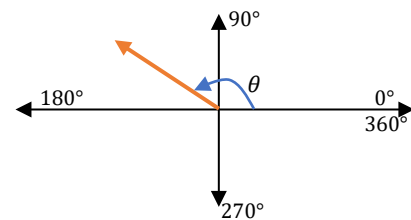
Section 4.1 The Unit Circle

An angle θ in standard form has its initial side on the positive x-axis (measured counterclockwise from 0°) to the terminal side of θ .

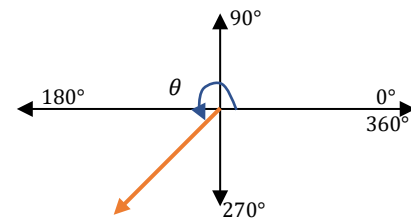
Quadrant I contains angles between _____



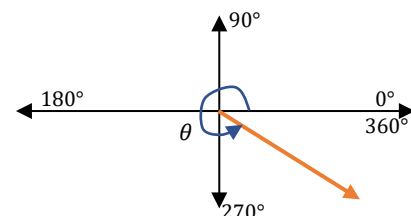
Quadrant II contains angles between _____



Quadrant III contains angles between _____



Quadrant IV contains angles between _____



Converting from Degrees to Radians and vice versa

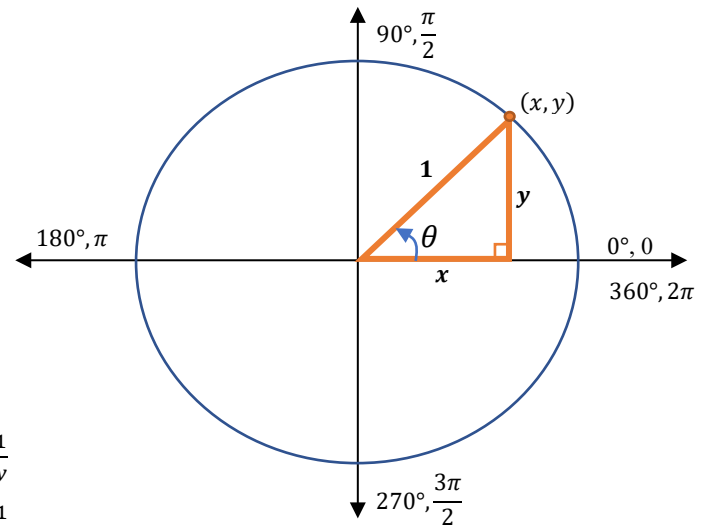
To change from degrees to radians- multiply the degree by $\frac{\pi}{180}$

To change from radians to degrees- multiply the radian by $\frac{180}{\pi}$

Exercise 1: Find the corresponding degree or radian

Degree	Radian
30°	
45°	
	$\frac{\pi}{3}$
	$\frac{\pi}{2}$
120°	
	$\frac{3\pi}{4}$
	$\frac{5\pi}{6}$
	π
210°	
	$\frac{5\pi}{4}$
240°	
270°	
	$\frac{5\pi}{3}$
315°	
330°	
360°	

Defining Trigonometric functions on the Unit Circle:



$$\sin \theta = y$$

$$\cos \theta = x$$

$$\tan \theta = \frac{y}{x}$$

$$\csc \theta = \frac{1}{y}$$

$$\sec \theta = \frac{1}{x}$$

$$\cot \theta = \frac{x}{y}$$

The Reciprocal Identities

$$\sin \theta = \frac{1}{\csc \theta}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

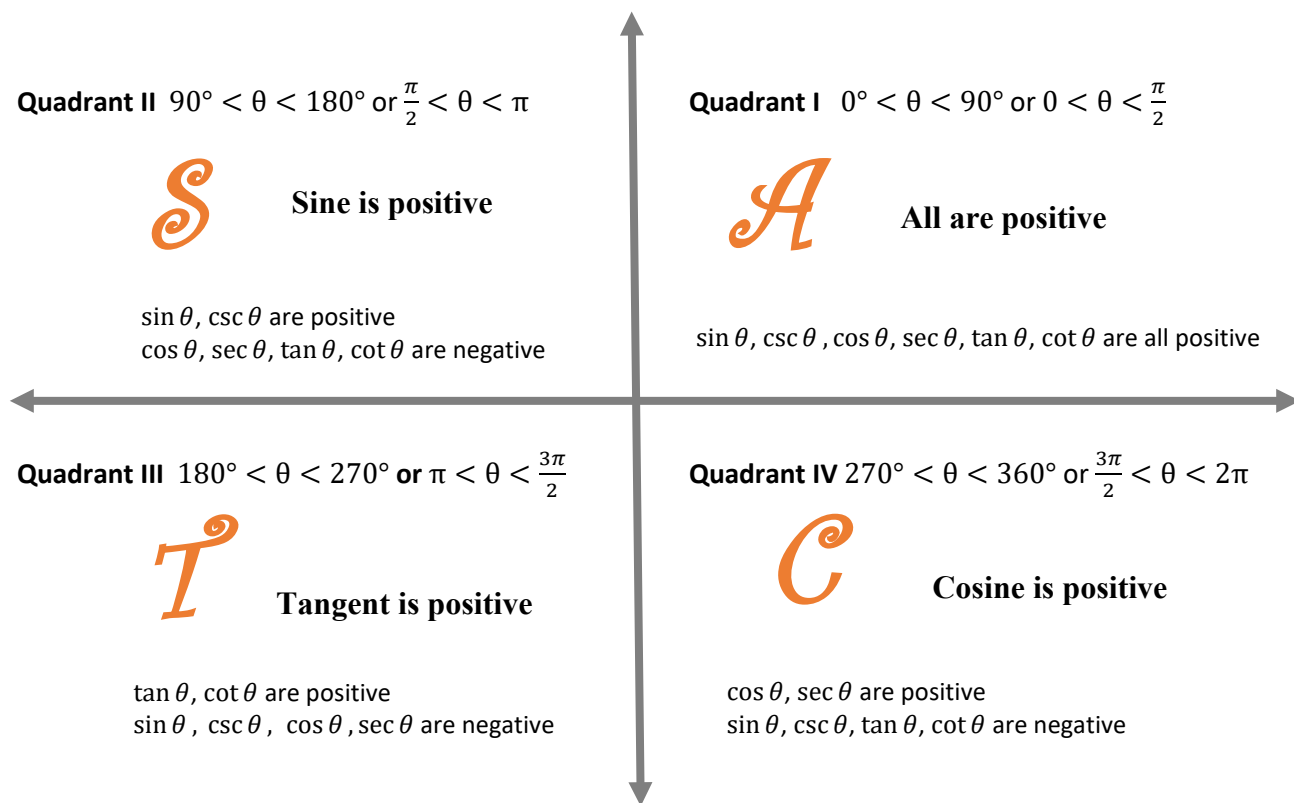
$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

The Quotient Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

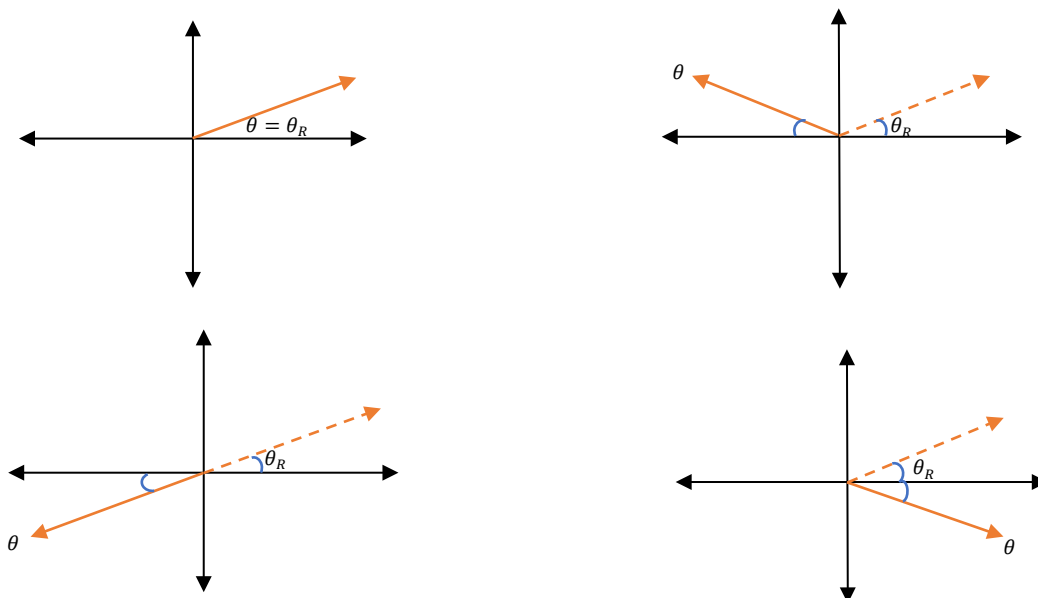
$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$



Reference Angle

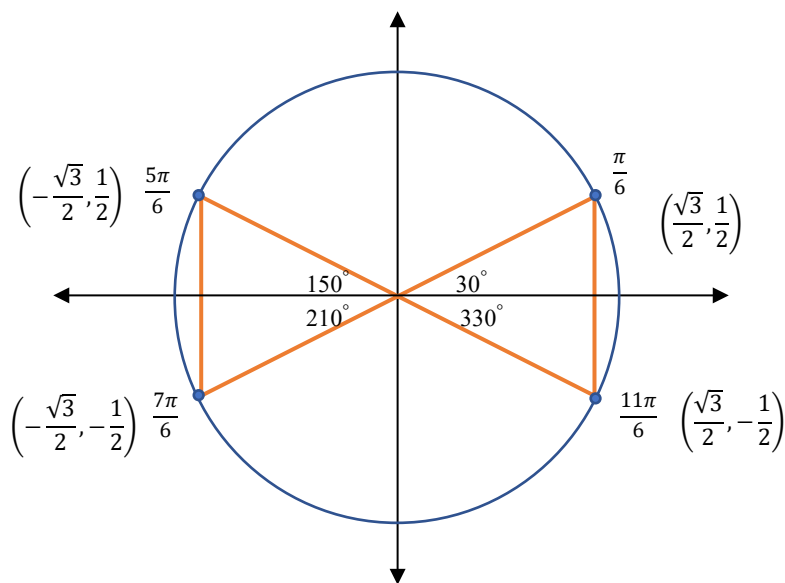
The **reference angle** θ_R for any angle θ in standard position is the positive acute angle between the terminal side of θ and the x-axis.

Angle and its reference angle:



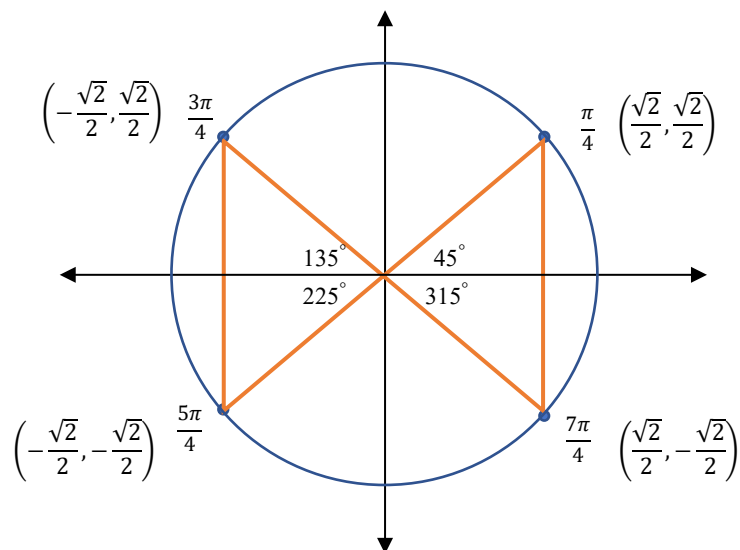
Use a bow-tie to relate angles with 30° or $\frac{\pi}{6}$ reference angle.

What can you say about their coordinates on the unit circle? _____



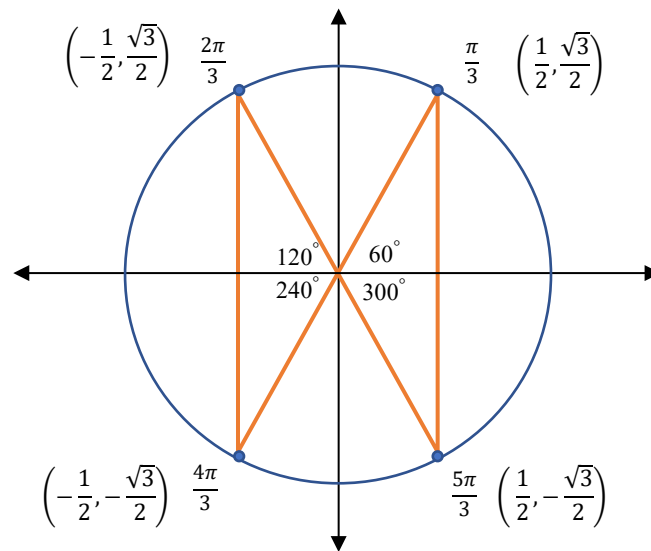
Use a bow-tie to relate angles with 45° or $\frac{\pi}{4}$ reference angle

What can you say about their coordinates on the unit circle? _____



Use a bow-tie to relate angles with 60° or $\frac{\pi}{3}$ reference angle

What can you say about their coordinates on the unit circle? _____



Try both Desmos activities below to see how the coordinates on the unit circle change with angles.



Desmos: The Unit Circle 1

<https://www.desmos.com/calculator/bydhfz5b9d>

Desmos: The Unit Circle 2

<https://www.desmos.com/calculator/p6cy3bxkkn>

The Unit Circle and the Special Angles

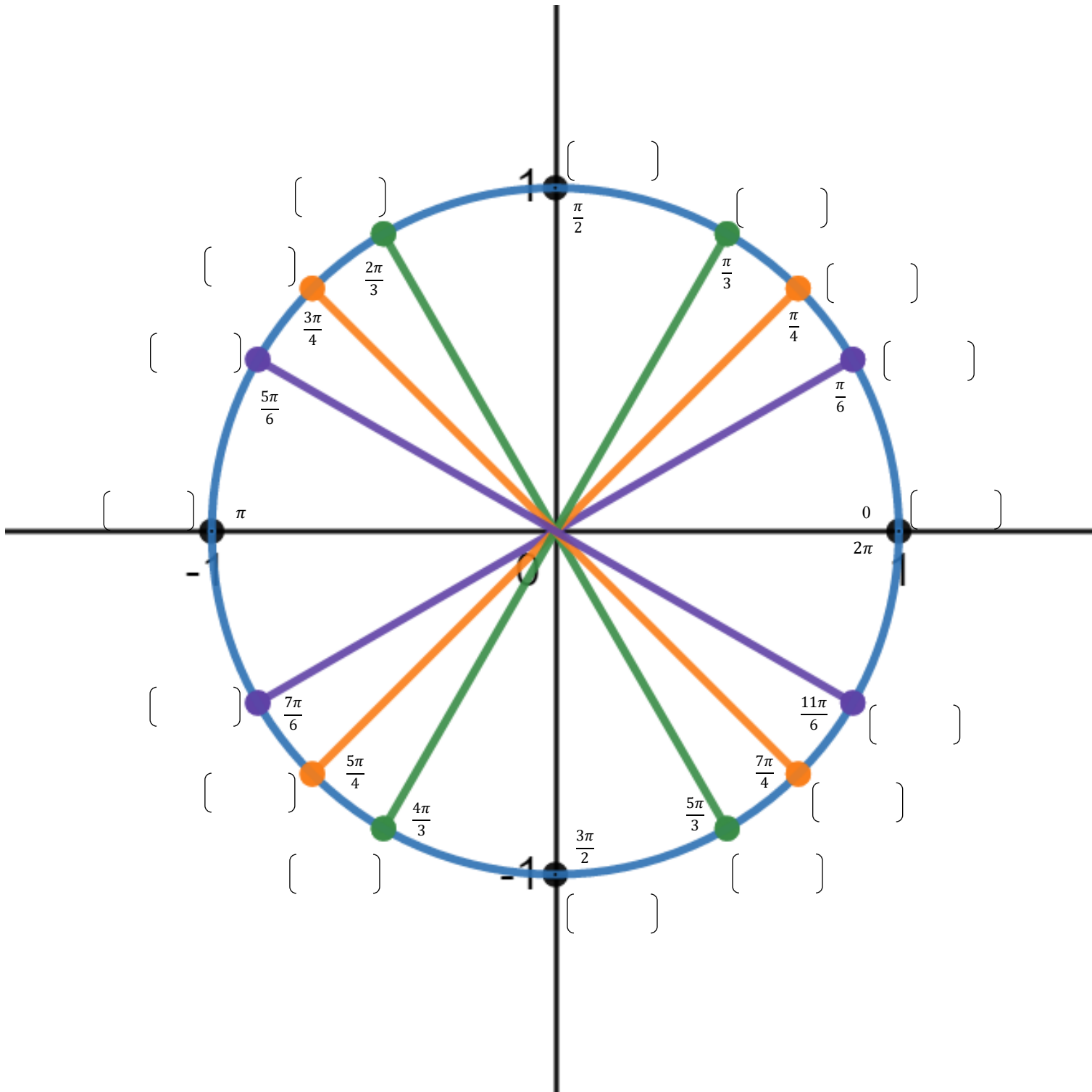
Color code the reference angles:

Purple: Reference angle 30°

Orange: Reference angle 45°

Green: Reference angle 60°

Exercise 2: Write the coordinates of the points on the unit circle in the brackets below.



Exercise 3: Find the value of the given trigonometric function without using a calculator.

a) $\tan 240^\circ$

b) $\sin \frac{5\pi}{6}$

c) $\sec 135^\circ$

d) $\cot \frac{2\pi}{3}$

e) $\csc 300^\circ$

f) $\cos \frac{7\pi}{4}$

Exercise 4: Find the value of the trigonometric functions of other special angles.

a) $\csc 0^\circ$

b) $\tan \pi$

c) $\sec 0^\circ$

d) $\cot \frac{\pi}{2}$

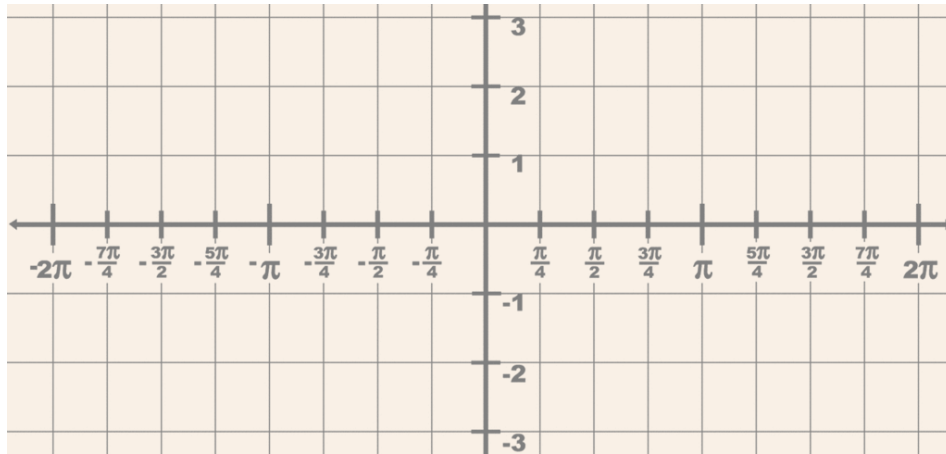
e) $\cot 180^\circ$

f) $\sec \frac{3\pi}{2}$

Section 4.2 Graphs of Sine and Cosine Functions

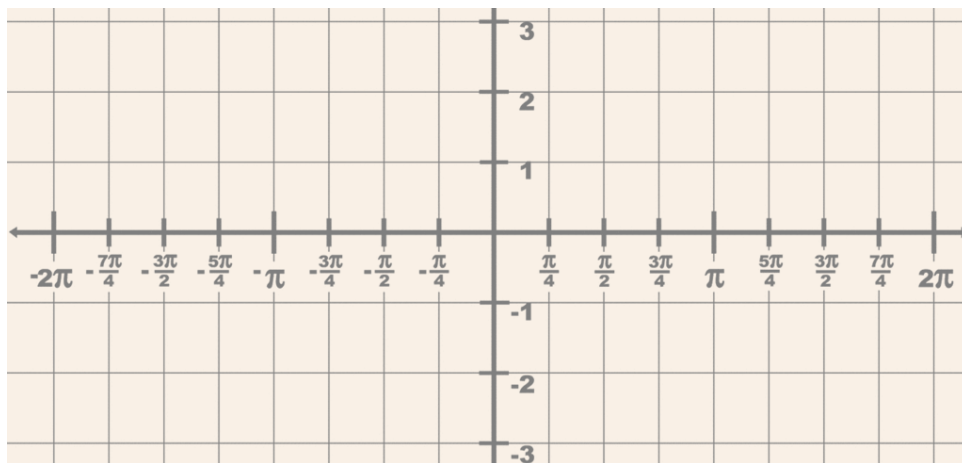
Fill out the table by determining the value of $y = \sin x$ for each given x . Plot the points on the graph.

x	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$y = \sin x$								



Fill out the table by determining the value of $y = \cos x$ for each given x . Plot the points on the graph.

x	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$y = \cos x$								



Characteristics of Sine and Cosine Graphs:

$$y = A \sin(Bx + C) + D$$

$$y = A \cos(Bx + C) + D$$

$$\text{Amplitude} = A$$

$$\text{Period} = \frac{2\pi}{B}$$

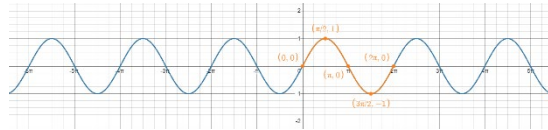
$$\text{Horizontal Phase Shift} = -\frac{C}{B}$$

$$\text{Vertical Shift} = D$$



Sine Graph

Desmos activity: Go to link <https://www.desmos.com/calculator/aeouweh9xr>



What happens to the sine graph when you change the amplitude A ?

What happens to the period of the sine graph when you change the value of B ?

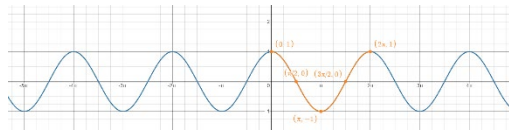
What happens to the sine graph when you change the vertical shift D ?

What happens to the sine graph when you change the value of C ?



Cosine Graph

Desmos activity: Go to link <https://www.desmos.com/calculator/iozygyep5a>



How is cosine graph similar to sine graph?

How is cosine graph different from sine graph?



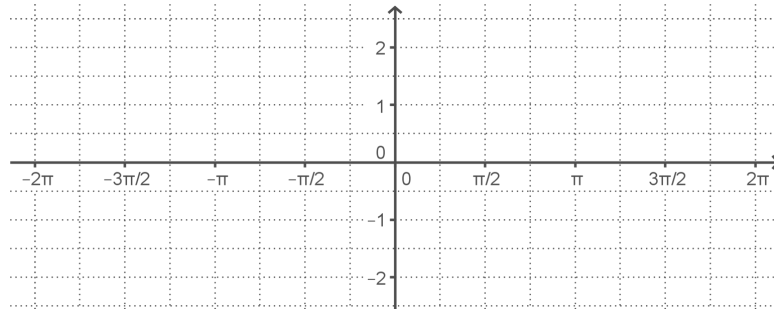
Phase Shift

Desmos activity: Go to link <https://www.desmos.com/calculator/rtn2hns4vm>

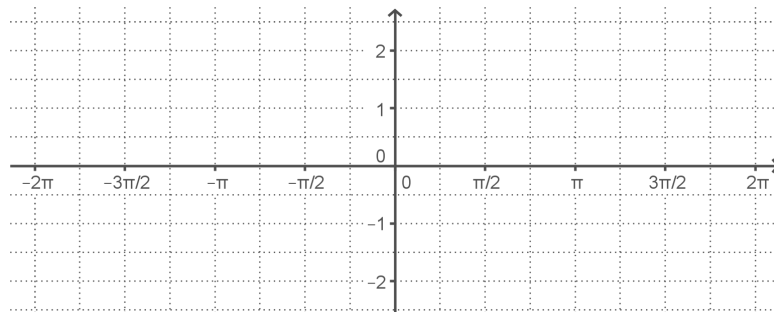
How are sine and cosine graphs related by phase shift?

Sketch **one-period** of each of the following functions:

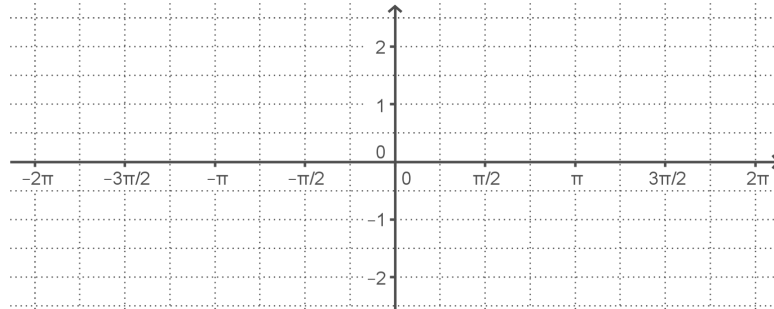
$$y = 2 \sin x$$



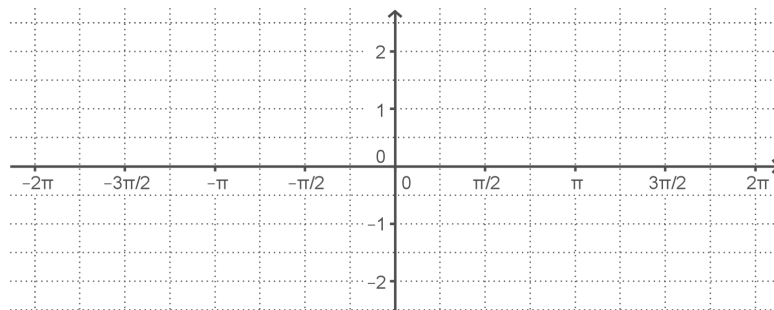
$$y = \sin 2x$$



$$y = -\cos x$$



$$y = \cos\left(x + \frac{\pi}{2}\right)$$



Section 4.3 Solving Trigonometric Equations

Example 1: Find all solutions in degrees in the interval $[0^\circ, 360^\circ)$ and in radians in the interval $[0, 2\pi)$.
The solutions are all angles that satisfy the given trigonometric value.

a) Solve for x : $\sin x = \frac{\sqrt{3}}{2}$.

b) Solve for x : $\cos x = -\frac{\sqrt{2}}{2}$.

c) Solve for x : $\tan x = -1$.

The process of solving trigonometric equations is very similar to the process of solving algebraic equations.

Step 1: Solve for the trigonometric function by carrying out algebraic procedures such as adding, subtracting, multiplying, dividing, factoring, or the square root property

Step 2: Identify the angles that correspond to the values of the trigonometric function

Find all solutions of the given equation in degrees in the interval $[0^\circ, 360^\circ)$ and in radians in the interval $[0, 2\pi)$.

Example 1: Solve a trigonometric equation	Solve a corresponding algebraic equation
$2 \sin x - 1 = 0$	$2x - 1 = 0$
Solve for $\sin x$: $2 \sin x - 1 = 0$ $2 \sin x = 1$ $\sin x = \frac{1}{2}$	Solve for x: $2x - 1 = 0$ $2x = 1$ $x = \frac{1}{2}$
Solutions are all angles that satisfy $\sin x = \frac{1}{2}$	
Solutions in degrees: $x = 30^\circ, 150^\circ$	
Solutions in radians: $x = \frac{\pi}{6}, \frac{5\pi}{6}$	

<p>Example 2: Solve a trigonometric equation</p> $\tan^2 x - 1 = 0$	<p>Solve a corresponding algebraic equation</p> $x^2 - 1 = 0$
<p>Solve for $\tan x$: $\tan^2 x - 1 = 0$</p> $(\tan x - 1)(\tan x + 1) = 0$ $\tan x - 1 = 0 \text{ or } \tan x + 1 = 0$ $\tan x = 1 \text{ or } \tan x = -1$ <p>Solutions are all angles that satisfy</p> $\tan x = 1 \text{ or } \tan x = -1$ <p>Find the solutions in degrees:</p> <p>Find the solutions in radians:</p>	<p>Solve for x:</p> $x^2 - 1 = 0$ $(x - 1)(x + 1) = 0$ $x - 1 = 0 \text{ or } x + 1 = 0$ $x = 1 \text{ or } x = -1$

Exercise 1: Find all solutions of the following equations in degrees in the interval $[0^\circ, 360^\circ)$ and in radians in the interval $[0, 2\pi)$.

a) Solve: $2 \cos x + 1 = 0$

b) Solve: $\sqrt{3} \tan x - 1 = 0$

c) Solve: $5 \sin x - \sqrt{3} = 3 \sin x$

d) Solve: $4 \sin^2 x - 1 = 0$

e) Solve: $(2 \cos x - \sqrt{3})(2 \cos x - 1) = 0$

SECTION 4.1 SUPPLEMENTARY EXERCISES

Find the value of the given trigonometric function without using a calculator.

a) $\cos 150^\circ$

i) $\cot(-210^\circ)$

b) $\sin 240^\circ$

j) $\csc(-60^\circ)$

c) $\tan 225^\circ$

k) $\sec(-135^\circ)$

d) $\csc 300^\circ$

l) $\sin 480^\circ$

e) $\cot 120^\circ$

m) $\tan 405^\circ$

f) $\sec 135^\circ$

n) $\csc 90^\circ$

g) $\cos(-240^\circ)$

o) $\cot 180^\circ$

h) $\tan(-150^\circ)$

p) $\sec(-90^\circ)$

SECTION 4.3 SUPPLEMENTARY EXERCISES

Find all solutions of the following equations in degrees in the interval $[0^\circ, 360^\circ)$ and in radians in the interval $[0, 2\pi)$

a) Solve: $2 \cos x - 1 = 0$

h) Solve: $5 \tan^2 x - 5 = 0$

b) Solve: $2 \sin x + 1 = 0$

i) Solve: $3(\sin x + 2) = 3 - \sin x$

c) Solve: $7 \tan x - 7 = 0$

j) Solve: $(3 \tan x + 1)(\tan x - 2) = 0$

d) Solve: $\sqrt{3} \tan x - 1 = 0$

k) Solve: $4(\cot x + 1) = 2(\cot x + 2)$

e) Solve: $5 \cos x - \sqrt{3} = 3 \cos x$

l) Solve: $3 \cos^2 x - 4 \cos x + 1 = 0$

f) Solve: $2 \sin x = 6 \sin x - \sqrt{12}$

m) Solve: $3 \sin^2 x + 7 \sin x + 2 = 0$

g) Solve: $4 \cos^2 x - 1 = 0$

n) Solve: $2 \cot^2 x - 13 \cot x + 6 = 0$

Answer Keys

Module 1

The Greatest Common Factor

- a) $5p(p^2 + 3p - 6)$
- b) $3ab^2(3a^2b^2 - 2ab + 1)$
- c) $(x + y)(5 - 6x)$

Factoring by Grouping

- a) $(8 + 3k)(7 + h)$
- b) $(x+8)(5x-y)$

Factoring Trinomials

Factoring Trinomials with Lead Coefficient 1 Page:7

- a) $(x - 2)(x - 6)$
- b) $(y - 3)(y - 12)$
- c) $(x + 5y)(x + 9y)$
- d) $(a - b)(a + 8b)$

Factoring Trinomials with Lead Coefficients other than 1 Page:8

- a) $(2h + 1)(h - 3)$
- b) $(h + 1)(2h - 3)$
- c) $(2h + 3)(h - 4)$
- d) $(k - 5)(3k + 1)$

Factoring Special Products

Difference of two Squares:

- a) $(x + 7)(x - 7)$
- b) $(5 + b)(5 - b)$
- c) $(6x + 4y)(6x - 4y)$
- d) $(4x^2 + 9)(2x - 3)(2x + 3)$
- e) $(10t + 7r)(10t - 7r)$
- f) It's not factorable

The sum of squares is not factorable.

Square of a Binomial Page: 9

- a) $(p + 5)^2$
- b) $(3b + 7)^2$

c) $(4a - 5)^2$

d) $(4y - 9)^2$

Sum or Difference of Two Cubes

a) $(x + 2)(x^2 - 2x + 4)$

b) $(3y - 1)(9y^2 + 3y + 1)$

c) $(4x - 3y)(16x^2 + 12xy + 9y^2)$

Cube of a Binomial Page:9

a) $(x - 1)^3$

b) $(x + 2)^3$

c) $(x - 3)^3$

Mixed Factoring/Factoring Completely Page: 11

a) $3x^2(x + 3)(x - 4)$

b) $5a(2 - a)(2 + a)$

c) $ab(4a^2 + 1)(2a + 1)(2a - 1)$

d) $2(2x - 3)(2x - 3)$

e) $xy(4x - 5y)^2$

f) $x(4x + 1)(3x + 2)$

g) $w^2(z + 4)(z - 14)$

Radicals and Complex Numbers Page:12**Simplifying Radicals**

Exercise 1:

a) 3

b) $3\sqrt{2}$

c) 5

d) $5\sqrt{3}$

e) 7

f) $7\sqrt{2}$

Exercise 2:

a) x

b) $x\sqrt{x}$

c) x^2

d) $x^2\sqrt{x}$

- e) x^3
- f) $x^3\sqrt{x}$
- g) x^{49}
- h) $x^{49}\sqrt{x}$

Exercise 3: Simplify each radical

- a) $2x^3y\sqrt{3y}$
- b) $6x^2y^3z^4\sqrt{5xz}$

Exercise 4: Rationalize denominators

- a) $\frac{9\sqrt{7}}{7}$
- b) $\frac{3\sqrt{x}}{2x}$
- c) $-2\sqrt{10} - 2\sqrt{6}$
- d) $\frac{5\sqrt{6}+2\sqrt{3}}{23}$

Complex Numbers

Exercise 5: Simplify the expressions

- a) $9i$
- b) $5i\sqrt{3}$
- c) $-7i$
- d) $i\sqrt{15}$

Powers of i :

$i^1 = i$	$i^5 = i$	$i^9 = i$
$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$
$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$
$i^4 = 1$	$i^8 = 1$	$i^{12} = 1$

Exercise 6: Simplify the product or quotient in terms of i

- a) $12i$
- b) 2
- c) $-6\sqrt{2}$
- d) $-3i\sqrt{2}$

Exercise 7: Perform the indicated operation

- a) $-3 + 6i$
- b) $31 + 29i$
- c) 13

Exercise 8: Rationalize the denominator

- a) $\frac{1}{2} - \frac{5}{2}i$

b) $\frac{-2}{7} + \frac{5}{7}i$

Section 1.3 Methods for Solving Quadratics

Method 1: Solving Quadratic Equations by Factoring

Exercise 1: Solve the equations by factoring Page:15

a) $x = -4, x = 8$

b) $y = -\frac{6}{7}, y = \frac{6}{7}$

c) $a = 0, a = 1, a = 5$

d) $x = -5, x = 0, x = \frac{1}{2}$

Method 2: Solving Quadratic Equations by the Square Root Property

Exercise 1: Solve using the square root property

a) $x = -10, x = 10$

b) $s = -2\sqrt{3}, s = 2\sqrt{3}$

c) $s = -2i\sqrt{6}, s = 2i\sqrt{6}$

d) $y = 3 - 7\sqrt{2}, y = 3 + 7\sqrt{2}$

e) $x = -1 - 2i\sqrt{6}, x = -1 + 2i\sqrt{6}$

Method 3: Solving Quadratic Equations by the Quadratic Formula

Exercise 1: Solve using the quadratic formula

a) $x = \frac{5-\sqrt{57}}{2}, x = \frac{5+\sqrt{57}}{2}$

b) $x = -1 - i\sqrt{5}, x = -1 + i\sqrt{5}$

c) $x = \frac{2-i\sqrt{6}}{2}, x = \frac{2+i\sqrt{6}}{2}$

Exercise 2: Page:18

- Factoring. It has one real (double) root.
- Square root property. It has two complex roots.
- The quadratic formula. It has two real roots.

Module II Polynomial Functions

Section 2.1

Exercise 1: Determine if each of the following sets forms a function

$\{(1,1),(2,1),(3,1)\}$: Each input x in the domain has exactly one output, it's a function. Domain: $[1, 3]$

$\{(1,1),(1,2),(1,3)\}$: One input x has more than one output, it's not a function.

Fahrenheit-Celsius conversion formula is a function because each input x in the domain has exactly one output. Domain: $(-\infty, \infty)$

Equation of a line: $y = mx + b$ is a function. Domain: $(-\infty, \infty)$

Circle is not a function.

Section 2.2 Quadratic Functions

Exercise 1:

- The vertex: $(0, 0)$, The Domain: $(-\infty, \infty)$, The Range: $[0, \infty)$, The roots: $(0, 0)$
- The vertex: $(0, -3)$, The Domain: $(-\infty, \infty)$, The Range: $[-3, \infty)$, The roots: $(-\sqrt{3}, 0)$, $(\sqrt{3}, 0)$
- The vertex: $(3, 0)$, The Domain: $(-\infty, \infty)$, The Range: $[0, \infty)$, The roots: $(3, 0)$

Exercise 2:

- The vertex: $(5, 6)$, The Domain: $(-\infty, \infty)$, The Range: $(-\infty, 6]$, The roots: $(-5 - \sqrt{6}, 0)$, $(-5 + \sqrt{6}, 0)$
- The vertex: $(-2, -8)$, The Domain: $(-\infty, \infty)$, The Range: $[-8, \infty)$, The roots: $(-4, 0)$, $(0, 0)$
- The vertex: $(0, -2)$, The Domain: $(-\infty, \infty)$, The Range: $(-\infty, -2]$, It doesn't have roots.

Exercise 3:

- The vertex: $(-2, -9)$, The Domain: $(-\infty, \infty)$, The Range: $[-9, \infty)$, The roots: $(-5, 0)$, $(1, 0)$
- The vertex: $(-1, -4)$, The Domain: $(-\infty, \infty)$, The Range: $(-\infty, -4]$, The function doesn't have roots.
- The vertex: $(1, 4)$, The Domain: $(-\infty, \infty)$, The Range: $[4, \infty)$, The function doesn't have roots.

Section 2.3 Polynomial Functions

Exercise 1:

The quotient: $3x + 4$, The remainder: 33

Exercise 2:

The quotient: $x^2 - 4x + 3$, The remainder: 0, The remainder being zero tells that -2 is a root of the function.

Factored form: $(x + 2)(x - 3)(x - 1)$

Roots: $(-2, 0)$, $(3, 0)$, $(1, 0)$. The roots are rational.

Exercise 3:

Rational root: $(-2, 0)$

Remaining roots: $x = 0 - 4i$, $x = 0 + 4i$. The roots are complex.

Factored form: $f(x) = (x + 2)(x - 4i)(x + 4i)$

Exercise 4:

A rational root: $(3,0)$

Remaining two roots: $(-1 - \sqrt{5}, 0), (-1 + \sqrt{5}, 0)$. They're irrational.

Exercise 5:

Rational roots: $(1,0), (-2,0)$

Remaining roots: $x = -i, x = i$ These roots are complex.

Exercise 6:

The polynomial in factored form: $f(x) = \frac{2}{3}(x + 3)(x - 1)(x - 2)$

Exercise 7:

The polynomial: $f(x) = x^4 - 6x^3 + 13x^2 - 24x + 36$

Module III Exponential and Logarithmic Functions

Exponential Functions and Graphs

Example 1:

x	$y = f(x) = 2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

Domain : $(-\infty, \infty)$ The range: $(0, \infty)$

Example 2:

x	$g(x) = \left(\frac{1}{2}\right)^x$	$h(x) = 2^{-x}$
-3	8	8
-2	4	4
-1	2	2
0	1	1
1	$\frac{1}{2}$	$\frac{1}{2}$

2	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{8}$	$\frac{1}{8}$

The Domain of two functions: $(-\infty, \infty)$ The range: $(-\infty, 0)$

Example 3:

x	$y = k(x)$ $= -2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	-1
1	-2
2	-4
3	-8

Domain: $(-\infty, \infty)$

Range: $(-\infty, 0)$

Section 3.2 Logarithmic Functions

Exercise 1:

Evaluate the logarithm	Rewrite in exponential form
$\log_3 9 = ?$	$\log_3 9 = 2 \Leftrightarrow 3^2 = 9$
$\log_3 1 = 0$	$3^0 = 1$
$\log_3 \frac{1}{9} = -2$	$3^{-2} = \frac{1}{9}$
$\log_2 8 = 3$	$2^3 = 8$

$\log_8 2 = 1/3$	$8^{\frac{1}{3}} = 2$
$\log_{\frac{1}{2}} 2 = -1$	$\left(\frac{1}{2}\right)^{-1} = 2$

Exercise 2:

- a) 0 b) 1 c) 2 d) 3 e) -1 f) -2 g) -3

Exercise 3:

x	y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$ Exercise 4: $\log_3 x$

x	y
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

Section 3.3 Properties of Logarithms

Exercise 1:

- a) 1.892

- b) -0.526
- c) 2.209
- d) 0.474
- e) 2.418

Exercise 2:

- a) $\log_3 9 + \left(\frac{1}{2}\right) \log_3 x - 2 \log_3 y$
- b) $\log(x + 3) - 2$
- c) $\ln(6) + 4 - \ln(4)$

Exercise 3:

- a) $\log(8)$
- b) $\log \frac{a^9}{16}$
- c) $\ln \frac{x^7(x+4)^6}{y^3}$

Exercise 4:

- a) 8
- b) $2x$
- c) $3x + y$
- d) x^2

Solving Exponential and Logarithmic 4.3

Exercise 1:

- a) $x = 2$
- b) $x = \frac{1}{2}$
- c) $x = -2$
- d) $x = 0$

Exercise 2:

- a) $x \approx 2.3219$
- b) $x \approx 0.8456$

Exercise 3:

- a) $x = 36$
- b) $x = 2$

c) $x \approx 1.6113$

d) $x = 2$

e) $x = 20$

Module IV Trigonometric Functions

Converting from Degrees to Radians and vice versa

Exercise 1:

Degree	Radian
30°	$\pi/6$
45°	$\pi/4$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
120°	$\frac{2\pi}{3}$
135°	$\frac{3\pi}{4}$
150°	$\frac{5\pi}{6}$
180°	π
210°	$\frac{7\pi}{6}$
225°	$\frac{5\pi}{4}$
240°	$\frac{4\pi}{3}$
270°	$\frac{3\pi}{2}$
300°	$\frac{5\pi}{3}$
315°	$\frac{7\pi}{4}$
330°	$\frac{11\pi}{6}$
360°	2π

The Unit Circle and the Special Angles

Exercise 3:

a) $\sqrt{3}$ b) $\frac{1}{2}$ c) $-\sqrt{2}$ d) $-\frac{\sqrt{3}}{3}$ e) $-\frac{2\sqrt{3}}{3}$ f) $\frac{\sqrt{2}}{2}$

Exercise 4:

- a) Undefined b) 0 c) 1 d) 0 e) Undefined f) Undefined

Section 4.3 Solving Trigonometric Equations

Example 1:

- a) $x = 60^\circ, 120^\circ$, in radians: $x = \frac{\pi}{3}, \frac{2\pi}{3}$
b) $x = 135^\circ, 225^\circ$ in radians: $x = \frac{3\pi}{4}, \frac{5\pi}{4}$
c) $x = 135^\circ, 315^\circ$ in radians: $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

Exercise 1: Find all solutions of the following equations in degrees in the interval $[0, 360)$ and in radians in the interval $[0, 2\pi)$

- a) $x = 120^\circ, 240^\circ$, in radians: $x = \frac{2\pi}{3}, \frac{4\pi}{3}$
b) $x = 30^\circ, 300^\circ$, in radians: $x = \frac{\pi}{6}, \frac{11\pi}{6}$
c) $x = 60^\circ, 120^\circ$, in radians: $x = \frac{\pi}{3}, \frac{2\pi}{3}$
d) $x = 30^\circ, 150^\circ$, in radians: $x = \frac{\pi}{6}, \frac{5\pi}{6}$
e) $x = 60^\circ, 300^\circ$, in radians: $x = \frac{\pi}{3}, \frac{5\pi}{3}$