Supplemental Problems **MAT 3770**

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1. Use **Lagrange Multiplier** methods to find the stationary values of z:
   (a) z = xy, subject to x + 2y = 2
   (b) z = x(y+4), subject to x + y = 8

2. For all of the above problem (1(a), 1(b)), find whether a slight relaxation of constraint will increase or decrease the optimal value of z. At what rate?

3. Use the **Lagrange Multiplier** method to solve the following problem: Minimize $x_1^2 + x_2^2$ subject to $x_1 x_2 = 1$.

4. Consider the utility functions of the form $U = x_1^{\alpha_1} x_2^{\alpha_2}$. Given the budget constraint, $p_1x_1 + p_2x_2 = M$, show that the implied demand curves are

$$x_1^M = \frac{\alpha_1}{\alpha_1 + \alpha_2}\frac{M}{p_1}$$
$$x_2^M = \frac{\alpha_2}{\alpha_1 + \alpha_2}\frac{M}{p_2}$$

   Find λ^M and $U^*(x_1^M, x_2^M)$ and verify that $\lambda^M = \frac{\partial U^*}{\partial M}$

5. Minimize $f(x) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3$ subject to $x_1 - x_2 + 2x_3 = 2$. Next, using the **Lagrange Multipliers** as shadow prices, predict the value of the optimal objective function if the constraint is increased by δ amount. Compare your prediction with the precise solution when δ = 0.5.

6. Minimize $\sum_{i=1}^{5} x_i^2$ subject to $x_1 + 2x_2 + x_3 = 1$ and $x_3 - 2x_4 + x_5 = 6$ by using the Lagrange Multiplier method.

7. Use Lagrange Multiplier methods to find the stationary values of z:

   (a) z = x – 3y – xy, subject to x + y = 6

(b) $z = 7 - y + x^2$, subject to $x + y = 0$

8. For all of the above problem (7(a), 7(b)), find whether a slight relaxation of constraint will increase or decrease the optimal value of z. At what rate?

9. Consider the following system of linear constraints:

$$\begin{aligned} &2x_1 + x_2 \leq\leq 100 \\ &x_1 + x_2 \leq 80 \\ &x_1 \leq 40 \\ &x_1, x_2 \geq 0 \end{aligned}$$

Determine the extreme points of the feasible region.

10. Consider the following system of inequalities:

$$\begin{aligned} &x_1 + x_2 \leq 5 \\ &x_1 + 2x_2 \leq 6 \\ &x_1, x_2 \geq 0 \end{aligned}$$

Find the extreme points of the region defined by these inequalities. Does this set have any directions of unboundedness? Either prove that none exists or give an example of direction of unboundedness.

11. Minimize z given the following constraints:

$$\begin{aligned} &z = -x_1 - 2x_2 \\ \text{subject to } &-2x_1 + x_2 \leq 2 \\ &-x_1 + x_2 \leq 3 \\ &x_1, x_2 \geq 0 \end{aligned}$$

12. Maximize $z = x_1 + 2x_2$ subject to $x_1 - 2x_2 \geq 12$, $x_1 + x_2 \geq 5$, $-x_1 + 3x_2 \leq 3$, $6x_1 - x_2 \geq 12$, $x_1, x_2 \geq 0$.

13. Minimize $f(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ subject to $x_1 + 2x_2 \geq 2, x_1 - x_2 \geq -1, -x_1 \geq -3$;

Show the feasible region graphically and solve this problem **graphically**.

14. Maximize $z = 6x_1 - 3x_2$ subject to $2x_1 + 5x_2 \geq 10$, $3x_1 + 2x_2 \leq 40$, $x_1 \leq 15$, $x_2 \leq 15$. Draw the feasible region and solve the problem **graphically**.

15. Find all local minimizers and maximizers of f:

$$f(x_1, x_2) = \frac{1}{3}x_1^3 + \frac{1}{2}x_1^2 + 2x_1x_2 + \frac{1}{2}x_2^2 - x_2 + 9.$$

16. Consider the function f: $f(x_1, x_2) = 8x_1^2 + 3x_1x_2 + 7x_2^2 - 25x_1 + 31x_2 - 29$.
Find all stationary points of this function and determine whether they are local minimizers or maximizers. Does this function have a global minimizer or a maximizer?

17. <u>An oil transfer company Problem:</u>

Consider a situation in which we are hired as consultants for a small oil transfer company. The management desires a policy of minimum cost due to restricted tank storage policy. Minimize the costs associated with dispensing and holding the oil to maintain sufficient oil to satisfy demand while meeting the restricted tank storage space constraint by using **Lagrange Multiplier method.**

Assumptions: Many factors determine the total cost of transferring oil. For our model we include the following variables: holding costs of the oil in the storage tank, costs of the oil in the storage tank, withdrawal rate of the oil from the tank per unit time, the cost of the oil, and the size of the tank.

Model Formulation:

x_i = the amount of oil type i available, i = 1, 2
a_i = the cost of the oil of type i
b_i = withdrawal rate per unit time of oil type i
h_i = holding(storage) costs per unit time for oil type i
t_i = space in cubic feet to store one unit of oil type i
T = total amount of storage space available

Historical records have been studied and a formula has been derived that describes the system costs in terms of our variables. Our objective is to minimize the sum of the cost variables given a specified maximum storage capacity.

Minimize $f(x_1, x_2) = \left(\frac{a_1b_1}{x_1} + \frac{h_1x_1}{2}\right) + \left(\frac{a_2b_2}{x_2} + \frac{h_2x_2}{2}\right)$

Such that $g(x_1, x_2) = t_1x_1 + t_2x_2 = T$

Oil type i	a_i($)	b_i($)	h_i($)	$t_i(ft^3)$
1	9	3	0.50	2
2	4	5	0.20	4

$T = 24\ ft^3$

18. **Carpenter's Problem:**

A carpenter makes tables and bookcases. He is trying to determine how many each type of furniture he should make each week. He wishes to determine a weekly production schedule for tables and bookcases that maximizes his profits. It costs $5 and $7 to produce tables and bookcases respectively. The carpenter realizes that a net unit profit of $25/table and $30/bookcase. He is trying to determine how many of each piece of furniture he should make each week. He has up to 690 board ft. of lumber to devote weekly to the project and up to 120 hour of labor. He estimates that it requires 20 board ft. of lumber and 5 hours of labor to complete a table and 30 board ft. of lumber and 4 hours of labor to complete a bookcase. Solve this problem using a) **graphically** and, b) **simplex method**.

19. Minimize $F(x) = x_1^2 + 3x_1x_2$ subject to $c_1(x) = x_1 + 5x_2 - 1 = 0$. Use the Lagrangian multiplier to predict what the optimum value would be if the constraint were changed to $x_1 + 5x_2 - \frac{4}{3} = 0$. Solve the problem with this modified constraint and hence determine the accuracy of the prediction. Also, do similar calculations for the modified constraint $x_1 + 5x_2 - \frac{2}{3} = 0$.

20. Solve the linear program **graphically**:

Minimize $z = x_1 + 2x_2$

subject to
$$2x_1 + x_2 \geq 12$$
$$x_1 + x_2 \geq 5$$
$$-x_1 + 3x_2 \leq 3$$
$$6x_1 - x_2 \geq 12$$
$$x_1, x_2 \geq 0.$$

21. Use **simplex method** to solve the following problem:

Maximize $z = 3x_1 - 4x_2 + 5x_3 + x_4$

$$-3x_1 - 2x_2 + 6x_3 - 9x_4 \geq 0$$
$$2x_1 + 4x_2 + 8x_3 - 5x_4 \geq -3$$
$$-3x_1 - x_2 + 2x_3 + 4x_4 \leq 15$$
$$x_j \geq 0 \text{ for all } j = 1, 2, 3, 4$$

22. **Barbie and Ken problems**

A manufacturer sells Barbie and Ken dolls. The manufacturing prices for each Barbie and Ken are \$6 and \$6.50 respectively. The company is trying to determine how many of each type the company should make to maximize its profit. The company has 100,000 oz. of plastics, 35,000 oz. of cardboard, and 30,000 oz. of nylon to use. The company estimates that it requires 12 oz. of plastics and 4 oz. of cardboard to make a Barbie doll and 14 oz.of plastics and 4 oz. of cardboard to complete Ken. Moreover, each Barbie doll requires 5 oz. of nylon. Model this problem as a linear programming problem. Solve by using simplex method.

23. Minimize the cost function C, where P is the pressure and R is the recycle ration. Determine the optimal values of the pressure and recycle ration and minimum cost within this constraint by direct substitution and also, by using Lagrange Multiplier method.

Minimize $C = 1000P + \frac{4\times 10^9}{PR} + 2.5\times 10^5 R$ subject to the constraints PR = 9000.

24. **Assignment Problem**

Suppose that a company is planning to assign 3 people to 3 jobs. The jobs are accountant, budget director, and personal manager. The first two people have degrees in business, but the second has ten years of corporate experience, while the first is just out of school. The first and second both have some management experience, but in different department. The third person's degree was in anthropology. Based on this information, the personnel depertment has determined numerical values $\{c_{i,j}\}$ corresponding to each person's appropriateness for a particular job. Model it as an **interger programming** problem.

25. Maximize $z = 5x_1 + 8x_2$ subject to $x_1 + x_2 \leq 6$, $5x_1 + 9x_2 \leq 45$, $x_1, x_2 \geq$

0 and both are integers. Solve this as an **integer programming** problem.

26. Minimize $f(x, y) = x^2 + 3y^2 – 4x – 6y$. Find out the minimum value of $f(x, y)$. Use a Matlab code to create a surface plot of $f(x, y)$ for $–10 \leq x \leq 10$, $-10 \leq y \leq 10$. Does the graph support your conclusion that you have found in the first part?