Review of Calculus for MAT 2680 – Differential Equations

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1 Derivatives

1.1 Definition and Notation

Let $y = f(x)$.

(1) The derivative is defined to be $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

(2) All of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x))$$

(3) All of the following are equivalent notations for derivative evaluated at $x = a$.

$$f'(a) = y'(a) = y'\Big|_{x=a} = \frac{df}{dx}\Big|_{x=a} = \frac{dy}{dx}\Big|_{x=a}$$

(4) All of the following are equivalent notations for the second derivative

$$(f'(x))' = f''(x) = y'' = \frac{d^2 f}{dx^2} = \frac{d^2 y}{dx^2} = \frac{d}{dx^2}(f(x))$$

1.2 Interpretation of the Derivative

Let $y = f(x)$.

(1) $m = f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$ and the equation of the tangent line at $x = a$ is given by $y = f(a) + f'(a)(x - a)$.

(2) $f'(x)$ is the instantaneous rate of the change of $f(x)$.

(3) If $f(x)$ is the position of an object at time x, then $f'(x)$ is the velocity of the object and $|f'(x)|$ is the speed.

(4) If $f'(x) > 0$ for all x in an interval I, then $f(x)$ is increasing on the interval I.

(5) If $f'(x) < 0$ for all x in an interval I, then $f(x)$ is decreasing on the interval I.

(6) If $f'(x) = 0$ for all x in an open interval I, then $f(x)$ is constant on the interval I.

1.3 Basic Properties and Formulas

Let $f(x)$ and $g(x)$ be differentiable functions (the derivative exists) and c and n be constants.

(1) $(c \cdot f(x))' = c \cdot f'(x)$

(2) $(f(x) \pm g(x))' = f'(x) \pm g'(x)$

(3) Product rule: $(f \cdot g)' = f' \cdot g + f \cdot g'$

(4) Quotient rule: $\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g - f \cdot g'}{g^2}$

(5) Chain rule: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

1.4 Common Derivatives

(1) $(\text{constant})' = 0$

(2) Power rule: $(x^n)' = n \cdot x^{n-1}$

(3) $(e^x)' = e^x$. In general, $(a^x)' = a^x \cdot \ln(a)$.

(4) $(\ln(x))' = \frac{1}{x}$. In general, $(\log_a x)' = \dfrac{1}{x \ln(a)}$.

(5) $(\sin x)' = \cos x$

(6) $(\cos x)' = -\sin x$

(7) $(\tan x)' = \sec^2 x$

(8) $(\sec x)' = \sec x \tan x$

(9) $(\cot x)' = -\csc^2 x$

(10) $(\csc x)' = -\csc x \cot x$

(11) $(\sin^{-1} x)' = \dfrac{1}{\sqrt{1-x^2}}$

(12) $(\cos^{-1} x)' = -\dfrac{1}{\sqrt{1-x^2}}$

(13) $(\tan^{-1} x)' = \dfrac{1}{1+x^2}$

2 Integrals

2.1 Definitions

(1) Indefinite Integral: $\int f(x)\,dx = F(x) + C$, where $F(x)$ is an anti-derivative of $f(x)$ and C is a constant.

(2) Definite Integral: $\int_a^b f(x)\,dx$ is defined to be the signed area of the region bounded by $y = f(x)$, $x = a$, $x = b$ and x-axis.

(3) Fundamental Theorem of Calculus: If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

where $F(x)$ is an anti-derivative of $f(x)$.

2.2 Basic Properties and Formulas

Let $f(x)$ and $g(x)$ be differentiable functions (the derivative exists) and c and n be constants.

(1) $\displaystyle\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$

(2) $\displaystyle\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$

(3) $\displaystyle\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

(4) $\displaystyle\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

2.3 Common Integrals (compare with Common Derivatives)

(1) $\displaystyle\int k\,dx = kx + C$

(2) Power rule ($n \neq -1$): $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

(3) Power rule ($n = -1$): $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

(4) $\displaystyle\int \frac{1}{ax+b}\,dx = \frac{\ln|ax+b|}{a} + C$

(5) $\displaystyle\int e^x\,dx = e^x + C; \quad \int e^{kx}\,dx = \frac{1}{k}e^{kx} + C.$

(6) $\displaystyle\int \sin x\,dx = -\cos x + C;$
$\displaystyle\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$

(7) $\displaystyle\int \cos x\,dx = \sin x + C;$
$\displaystyle\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C$

(8) $\displaystyle\int \sec^2 x\,dx = \tan x + C$

(9) $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

(10) $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

(11) $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

(12) $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(13) $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

2.4 Standard Integration Techniques

- The following types of problems can be solved by integration by substitution: $\displaystyle\int f(x)\,dx = \int g(u)\,du$, where $u = u(x)$ and $du = u'(x)\,dx$.

(1) $\displaystyle\int x^2 \cos(x^3)\,dx$
Set $u = x^3$. Then $du = (x^3)'\,dx = 3x^2\,dx$ and

$$\int x^2 \cos(x^3)\,dx = \int \cos u \cdot \frac{1}{3}du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(x^3) + C.$$

(2) $\int x^2\sqrt{x^3+5}\,dx$

Set $u = x^3+5$. Then $du = (x^3+5)'\,dx = 3x^2\,dx$ and

$$\int x^2\sqrt{x^3+5}\,dx = \int \sqrt{u}\cdot\frac{1}{3}du = \frac{1}{3}\cdot\frac{u^{3/2}}{3/2} + C = \frac{2}{9}(x^3+5)^{3/2} + C.$$

- The following types of problems can be solved by integration by parts $\int u\,dv = uv - \int v\,du$.

(1) $\int (3x+1)e^{2x}\,dx$

Set $u = 3x+1$, $dv = e^{2x}\,dx$. Then $du = (3x+1)'\,dx = 3\,dx$ and $v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$.

$$\begin{aligned}\int (3x+1)e^{2x}\,dx &= \int (3x+1)\cdot e^{2x}\,dx = \int u\cdot dv = uv - \int v\,du \\ &= (3x+1)\cdot\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\cdot 3\,dx \\ &= \frac{1}{2}(3x+1)e^{2x} - \frac{3}{2}\int e^{2x}\,dx = \frac{1}{2}(3x+1)e^{2x} - \frac{3}{4}e^{2x} + C \\ &= \frac{3}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.\end{aligned}$$

(2) $\int x^3\ln x\,dx$

Set $u = \ln x$, $dv = x^3\,dx$. Then $du = (\ln x)'\,dx = \frac{1}{x}\,dx$ and $v = \int x^3\,dx = \frac{1}{4}x^4$.

$$\begin{aligned}\int x^3\ln x\,dx &= \int (\ln x)\cdot x^3\,dx = \int u\cdot dv = uv - \int v\,du \\ &= (\ln x)\cdot\frac{1}{4}x^4 - \int \frac{1}{4}x^4\cdot\frac{1}{x}\,dx \\ &= \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx \\ &= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C\end{aligned}$$

(3) $\int (2x+1)\sin(x)\,dx$

Set $u = 2x+1$, $dv = \sin x\,dx$. Then $du = (2x+1)'\,dx = 2\,dx$ and $v = \int \sin x\,dx = -\cos x$.

$$\begin{aligned}\int (2x+1)\sin(x)\,dx &= \int (2x+1)\cdot\sin x\,dx = \int u\cdot dv = uv - \int v\,du \\ &= (2x+1)\cdot(-\cos x) - \int(-\cos x)\cdot 2\,dx \\ &= -(2x+1)\cdot(\cos x) + 2\int \cos x\,dx \\ &= -(2x+1)\cdot(\cos x) + 2\sin x + C\end{aligned}$$

- The integration of rational functions can be solved by partial fraction decomposition.

(1) $\displaystyle\int \frac{3x+5}{(x-1)(x+3)}\,dx$

Find constants A, B so that $\dfrac{3x+5}{(x-1)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+3}$. Multiply the common denominator $(x-1)(x+3)$:

$$3 \cdot x + 5 = A(x+3) + B(x-1) = (A+B) \cdot x + (3A - B)$$

Compare coefficients:

$$\begin{cases} A + B = 3 \\ 3A - B = 5 \end{cases}$$

Solve the system of equation: $A = 2,\ B = 1$. Thus

$$\begin{aligned} \int \frac{3x+5}{(x-1)(x+3)}\,dx &= \int \frac{A}{x-1} + \frac{B}{x+3}\,dx = \int \frac{2}{x-1} + \frac{1}{x+3}\,dx \\ &= 2\ln|x-1| + \ln|x+3| + C \end{aligned}$$

(2) $\displaystyle\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx$

Find constants A, B, C so that

$$\frac{7x^2+13x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}.$$

The rest of the solution is similar to (1). For your reference, the answer is posted below.

$$\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx = 4\ln|x-1| + \frac{3}{2}\ln(x^2+4) + 8\tan^{-1}\left(\frac{x}{2}\right) + C$$