MAT 1575 Final Exam Review Problems

Revised by Prof. Kostadinov Spring 2014, Prof. ElHitti Summer 2017, Prof. Africk Spring 2023

1. Evaluate the following definite integrals:

   a. $\int_0^1 x^2(x^3+1)^3\,dx$ b. $\int_0^1 \frac{x}{\sqrt{x^2+9}}\,dx$ c. $\int_0^1 \frac{3x^2}{\sqrt[3]{x^3+1}}\,dx$

2. Evaluate the following indefinite integrals:

   a. $\int x^2\ln(x)\,dx$ b. $\int x^2e^{-x}\,dx$ c. $\int x\cos(3x)\,dx$

3. Find the area of the region enclosed by the graphs of:

   a. $y=3-x^2$ and $y=-2x$ b. $y=x^2-2x$ and $y=x+4$

4. Find the volume of the solid obtained by rotating the region bounded by the graphs of:
   a. $y=x^2-9$, $y=0$ about the x-axis.   b. $y=16-x$, $y=3x+12$, $x=-1$ about the x-axis.
   c. $y = x^2 + 2$, $y = -x^2 + 10$, $x \geq 0$ about the y-axis.

5. Evaluate the following indefinite integrals:

   a. $\int \frac{1}{x^2\sqrt{36-x^2}}\,dx$ b. $\int \frac{\sqrt{x^2-9}}{x^4}\,dx$ c. $\int \frac{9}{x^2\sqrt{x^2+9}}\,dx$ d. $\int \frac{6}{x^2\sqrt{x^2-36}}\,dx$

6. Evaluate the following indefinite integrals:

   a. $\int \frac{3x+7}{x^2+6x+9}\,dx$ b. $\int \frac{5x+6}{x^2-36}\,dx$ c. $\int \frac{3x+2}{x^2+2x-8}\,dx$

7. Evaluate the improper integral:

   a. $\int_0^\infty \frac{2}{(x+2)^3}\,dx$ b. $\int_0^\infty \frac{5}{\sqrt[5]{x+5}}\,dx$ c. $\int_3^5 \frac{3}{\sqrt[3]{(x-3)^4}}\,dx$

8. Decide if the following series converges or not. Justify your answer using an appropriate test:

   a. $\sum_{n=1}^{n=\infty} \frac{9n^5}{3n^5+5}$ b. $\sum_{n=1}^{\infty} \frac{5}{10^n}$ c. $\sum_{n=1}^{\infty} \frac{5n}{10^n}$ d. $\sum_{n=1}^{n=\infty} \frac{n!}{n^2\,5^n}$ e. $\sum_{n=1}^{n=\infty} \left(\frac{n+1}{2n+3}\right)^n$

9. Determine whether the series is absolutely or conditionally convergent or divergent:

   a. $\sum_{n=1}^{\infty}(-1)^n\frac{10}{7n+2}$ b. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^5}}$ c. $\sum_{n=0}^{\infty}(-1)^n 5^{-n}$ d. $\sum_{n=1}^{\infty}(-1)^n\frac{n^2-n-1}{2n^2+n+1}$

10. Find the radius and the interval of convergence of the following power series:

$a.\ \sum_{n=0}^{\infty}\frac{(x-1)^n}{n+2}$ $\quad b.\ \sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^n}{n+2}$ $\quad c.\ \sum_{n=1}^{\infty}\frac{(x+1)^n}{n5^n}$ $\quad d.\ \sum_{n=1}^{\infty}\frac{(-1)^n(x+1)^n}{n5^n}$

11. Find the Taylor polynomial of degree 2 for the given function, centered at the given number a:

a. $f(x)=e^{-2x}$ at $a=-1$. b. $f(x)=\cos(5x)$ at $a=2\pi$.

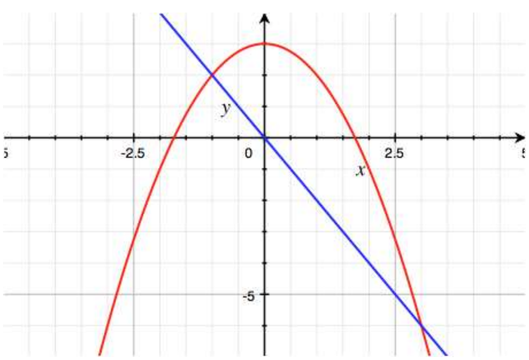
12. Find the Taylor polynomial of degree 3 for the given function, centered at the given number a:

a. $f(x)=1+e^{-x}$ at $a=-1$ b. $f(x)=\sin(x)$ at $a=\frac{\pi}{2}$

Answers:

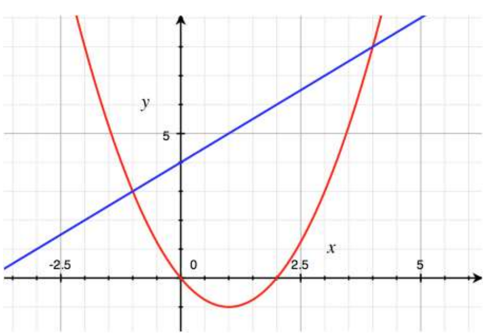
(1a). $\frac{5}{4}$ **(1b).** $\sqrt{10}-3$ **(1c).** $\frac{3}{2}\left(2^{2/3}-1\right)$

(2a). $\frac{x^3\ln(x)}{3}-\frac{x^3}{9}+C$ **(2b).** $-(x^2+2x+2)e^{-x}+C$ **(2c).** $\frac{1}{3}x\sin(3x)+\frac{1}{9}\cos(3x)+C$



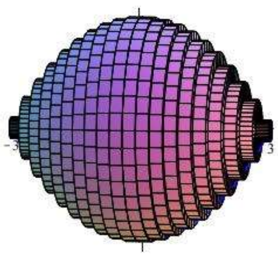
(3a). The area of the region between the two curves is:

$$Area=\int_{-1}^{3}\left(3-x^2-(-2x)\right)dx=\frac{32}{3}$$



(3b). The area of the region between the two curves is:

$$Area=\int_{-1}^{4}\left(x+4-\left(x^2-2x\right)\right)dx=\frac{125}{6}$$



(4a). Approximate the volume of the solid by vertical disks with radius $y = x^2 - 9$ between $x = -3$ and $x = 3$; gives the volume is $V = \int_{-3}^{3} \pi (x^2 - 9)^2\, dx = \frac{1296}{5}\pi$.

(4b). Using a washer of outer radius $R_{outer} = 16 - x$ and inner radius $R_{inner} = 3x + 12$ at x, gives the volume: $V = \pi \int_{-1}^{1} ((16-x)^2 - (3x+12)^2)dx = \frac{656\pi}{3}$, where the upper limit 1 is obtained from $16 - x = 3x + 12 \Rightarrow x = 1$.

(4c). 16π

(5a). $-\frac{\sqrt{36-x^2}}{36x} + C$ **(5b).** $\frac{(x^2-9)^{3/2}}{27x^3} + C$ **(5c).** $-\frac{\sqrt{x^2+9}}{x} + C$ **(5d).** $\frac{\sqrt{x^2-36}}{6x} + C$

(6a). $\frac{2}{x+3} + 3ln|x+3| + C$ **(6b).** $3\ln|x-6| + 2\ln|x+6| + C$ **(6c).** $\frac{5}{3}\ln|x+4| + \frac{4}{3}\ln|x-2| + C$

(7a). $\frac{1}{4}$ **(7b).** The integral does not converge **(7c).** The integral does not converge

(8a). $\lim_{n\to\infty} \frac{9n^5}{3n^5+5} = \lim_{n\to\infty} \frac{3}{1+5/3n^5} = 3 > 0$ so the series diverges by the nth term test for divergence.

(8b). This is a geometric series, with common ration r = 1/10 <1, so it converges to 5/9:

$$\sum_{n=1}^{\infty} \frac{5}{10^n} = \frac{a}{1-r} = \frac{5/10}{1-\frac{1}{10}} = \frac{5/10}{9/10} = \frac{5}{9}$$

(8c). $\lim_{n\to\infty} \frac{5(n+1)}{10^{n+1}} \Big/ \frac{5n}{10^n} = \lim_{n\to\infty} \frac{1+1/n}{10} = \frac{1}{10} < 1$ so the series converges by the ratio test.

(8d). $\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^2 5^{n+1}} \Big/ \frac{n!}{n^2 5^n} = \lim_{n\to\infty} \frac{n^2}{5(n+1)} = \lim_{n\to\infty} \frac{n}{5(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{n}{5} = \infty$ so the series diverges by the ratio test.

(8e). $\lim_{n\to\infty} [\left(\frac{n+1}{2n+3}\right)^n]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n+1}{2n+3} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{3}{n}} = \frac{1}{2} < 1$ so the series converges by the nth root test.

(9a). Conditionally convergent: The series converges by the alternating series test since

$\frac{10}{7n+2} > \frac{10}{7(n+1)+2}$ and $\lim_{n\to\infty} \frac{10}{7n+2} = 0$ but not absolutely since $\sum_{n=1}^{\infty} \left|(-1)^n \frac{10}{7n+2}\right| = \sum_{n=1}^{\infty} \frac{10}{7n+2}$ diverges by comparing it with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges, using the limit comparison test: $\lim_{n\to\infty} \frac{10}{7n+2} / \frac{1}{n} = \lim_{n\to\infty} \frac{10n}{7n+2} = \frac{10}{7} < \infty$.

(9b). Absolutely convergent: $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{\sqrt{n^5}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ a convergent p-series with p = 5/2 > 1.

(9c). Absolutely convergent: $\sum_{n=0}^{\infty} |(-1)^n 5^{-n}| = \sum_{n=0}^{\infty} 5^{-n}$ is a convergent geometric series with common ratio r = 1/5 < 1.

(9d). $\lim_{n\to\infty} \frac{n^2-n-1}{2n^2+n+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n}-\frac{1}{n^2}}{2+\frac{1}{n}+\frac{1}{n^2}} = \frac{1}{2} > 0$ so the series diverges by the nth term test for divergence.

(10a). The power series converges when $|x-1| < 1$ by the ratio test, which gives a radius of convergence 1 and interval of convergence centered at 1. The series diverges at x = 2 (harmonic series) but converges at x = 0 (alternate harmonic series), so the interval of convergence is $0 \le x < 2$.

(10b). The power series converges when $|x-1| < 1$ by the ratio test, which gives a radius of convergence 1 and interval of convergence centered at 1. The series diverges at x = 0 (harmonic series) but converges at x = 2 (alternate harmonic series), so the interval of convergence is $0 < x \le 2$.

(10c). The power series converges when $|x+1| < 5$ by the ratio test, which gives a radius of convergence 5 and interval of convergence centered at -1. The series diverges at x = 4 (harmonic series) but converges at $x = -6$ (alternate harmonic series), so the interval of convergence is $-6 \le x < 4$.

(10d). The power series converges when $|x+1| < 5$ by the ratio test, which gives a radius of convergence 5 and interval of convergence centered at -1. The series diverges at $x = -6$ (harmonic series) but converges at x = 4 (alternate harmonic series), so the interval of convergence is $-6 < x \le 4$.

(11a). $p_2(x) = e^2 - 2e^2(x+1) + 2e^2(x+1)^2$

(11b). $p_2(x) = 1 - \frac{25}{2}(x - 2\pi)^2$

(12a). $p_3(x) = 1 + e - e(x+1) + \frac{e}{2}(x+1)^2 - \frac{e}{6}(x+1)^3$

$= 1 + \frac{e}{3} - \frac{e}{2}x - \frac{e}{6}x^3$

(12b). $p_3(x) = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2$