MAT 1575 Final Exam Review Problems

Revised by Prof. Kostadinov Spring 2014, Prof. ElHitti Summer 2017, Prof. Africk Spring 2018

1. Evaluate the following definite integrals:
   a. \( \int_0^1 x^2 (x^3 + 1)^3 \, dx \)
   b. \( \int_0^\infty \frac{x}{\sqrt{x^2 + 9}} \, dx \)
   c. \( \int_0^\infty \frac{3x^2}{\sqrt[3]{x^3 + 1}} \, dx \)

2. Evaluate the following indefinite integrals:
   a. \( \int x^2 \ln(x) \, dx \)
   b. \( \int x^2 e^{-x} \, dx \)
   c. \( \int x \cos(3x) \, dx \)

3. Find the area of the region enclosed by the graphs of:
   a. \( y = 3 - x^2 \) and \( y = 2x \)
   b. \( y = x^2 \) and \( y = x + 4 \)

4. Find the volume of the solid obtained by rotating the region bounded by the graphs of:
   a. \( y = x^2 \) and \( y = 0 \) about the x-axis.
   b. \( y = 16 - x \) and \( y = 3x + 12 \) about the x-axis.
   c. \( y = x^2 + 2 \) and \( y = -x^2 + 10 \) about the y-axis.

5. Evaluate the following indefinite integrals:
   a. \( \int \frac{1}{x^2 \sqrt{36}} \, dx \)
   b. \( \int \frac{\sqrt{x^2}}{x^4} \, dx \)
   c. \( \int \frac{9}{x^2 \sqrt{x^2 + 9}} \, dx \)
   d. \( \int \frac{6}{x^2 \sqrt{x^2 - 36}} \, dx \)

6. Evaluate the following indefinite integrals:
   a. \( \int \frac{3x + 7}{x^2 + 6x + 9} \, dx \)
   b. \( \int \frac{5x + 6}{x^2} \, dx \)
   c. \( \int \frac{3x + 2}{x^2 + 2x} \, dx \)

7. Evaluate the improper integral:
   a. \( \int_0^\infty \frac{2}{(x + 2)^3} \, dx \)
   b. \( \int_0^\infty \frac{5}{\sqrt{x + 5}} \, dx \)
   c. \( \int_3^\infty \frac{3}{\sqrt{(x - 3)^4}} \, dx \)

8. Decide if the following series converges or not. Justify your answer using an appropriate test:
   a. \( \sum_{n=1}^{\infty} \frac{9n^5}{3n^5 + 5} \)
   b. \( \sum_{n=1}^{\infty} \frac{5}{10^n} \)
   c. \( \sum_{n=1}^{\infty} \frac{5n}{10^n} \)
   d. \( \sum_{n=1}^{\infty} \frac{n!}{n^2 5^n} \)
   e. \( \sum_{n=1}^{\infty} \left( \frac{n+1}{2n+3} \right)^n \)

9. Determine whether the series is absolutely or conditionally convergent or divergent:
   a. \( \sum_{n=1}^{\infty} \frac{10}{7n + 2} \)
   b. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^5}} \)
   c. \( \sum_{n=0}^{\infty} (-1)^n \frac{n}{5^n} \)
   d. \( \sum_{n=1}^{\infty} (-1)^n \frac{n^2 - n - 1}{2n^2 + n + 1} \)
10. Find the radius and the interval of convergence of the following power series:

\[ a. \sum_{n=0}^{\infty} \frac{(x - 1)^n}{n + 2} \quad b. \sum_{n=0}^{\infty} \frac{(-1)^n(x - 1)^n}{n + 2} \quad c. \sum_{n=1}^{\infty} \frac{(x + 1)^n}{n 5^n} \quad d. \sum_{n=1}^{\infty} \frac{(-1)^n(x + 1)^n}{n 5^n} \]

11. Find the Taylor polynomial of degree 2 for the given function, centered at the given number a:

a. \( f(x) = e^{-2x} \) at \( a = -1 \)

b. \( f(x) = \cos(5x) \) at \( a = \frac{2\pi}{2} \)

12. Find the Taylor polynomial of degree 3 for the given function, centered at the given number a:

a. \( f(x) = 1 + e^{-x} \) at \( a = -1 \)

b. \( f(x) = \sin(x) \) at \( a = \frac{\pi}{2} \)

Answers:

(1a). \( \frac{5}{4} \)  (1b). \( \sqrt{10} - 3 \)  (1c). \( \frac{3}{2} \left( \frac{2}{3} - 1 \right) \)

(2a). \( \frac{x^3 \ln(x)}{3} + C \)  (2b). \( (x^2 + 2x + 2)e^{-x} + C \)  (2c). \( \frac{1}{3} x \sin(3x) + \frac{1}{9} \cos(3x) + C \)

(3a). The area of the region between the two curves is:
\[
Area = \int_{1}^{5} \left( 3 - x^2 - (2x) \right) dx = \frac{32}{3}
\]

(3b). The area of the region between the two curves is:
\[
Area = \int_{1}^{4} \left( x + 4 - (x^2 - 2x) \right) dx = \frac{125}{6}
\]
(4a). Approximate the volume of the solid by vertical disks with radius \( y = x^2 - 9 \) between \( x = -3 \) and \( x = 3 \); gives the volume is
\[
V = \int_{-3}^{3} \pi (x^2 - 9)^2 \, dx = \frac{1296\pi}{5}.
\]

(4b). Using a washer of outer radius \( R_{outer} = 16 - x \) and inner radius \( R_{inner} = 3x + 12 \) at \( x \), gives the volume:
\[
V = \pi \int_{-1}^{1} ((16 - x)^2 - (3x + 12)^2) \, dx = \frac{656\pi}{3},
\]
where the upper limit 1 is obtained from \( 16 - x = 3x + 12 \Rightarrow x = -1 \).

(4c). 16π

(5a). \( \frac{\sqrt{36} \cdot x^2}{36x} + C \)
(5b). \( \frac{(x^2 - 9)^{3/2}}{27x^3} + C \)
(5c). \( \frac{\sqrt{x^2 + 9}}{x} + C \)
(5d). \( \frac{\sqrt{x^2 - 36}}{6x} + C \)

(6a). \( \frac{2}{x+3} + 3\ln|x + 3| + C \)
(6b). \( 3\ln|x| + 2\ln|x + 6| + C \)
(6c). \( \frac{5}{3}\ln|x| + 4 + \frac{4}{3}\ln|x + 2| + C \)

(7a). \( \frac{1}{4} \)
(7b). The integral does not converge
(7c). The integral does not converge

(8a). \( \lim_{n \to \infty} \frac{9n^5}{3n^5+5} = \lim_{n \to \infty} \frac{3}{1+5/3n^5} = 3 > 0 \) so the series diverges by the nth term test for divergence.

(8b). This is a geometric series, with common ratio \( r = \frac{1}{10} < 1 \), so it converges to \( \frac{5}{9} \):
\[
\sum_{n=1}^{\infty} \frac{5}{10^n} = \frac{a}{1-r} = \frac{5/10}{1-1/10} = \frac{5}{9/10} = \frac{5}{9}.
\]

(8c). \( \lim_{n \to \infty} \frac{5(n+1)}{10^{n+1}} / \frac{5n}{10^n} = \lim_{n \to \infty} \frac{1+1/5n}{10} = \frac{1}{10} < 1 \) so the series converges by the ratio test.

(8d). \( \lim_{n \to \infty} \frac{(n+1)!}{(n+1)^2 5^{n+1}} / \frac{n!}{n^2 5^n} = \lim_{n \to \infty} \frac{n^2}{5(n+1)} = \lim_{n \to \infty} \frac{n}{5(1+\frac{1}{n})} = \lim_{n \to \infty} \frac{n}{5} = \infty \) so the series diverges by the ratio test.

(8e). \( \lim_{n \to \infty} \left( \frac{n+1}{2n+3} \right)^n = \lim_{n \to \infty} \frac{n+1}{2n+3} = \lim_{n \to \infty} \frac{1+\frac{1}{n}}{2+\frac{3}{n}} = \frac{1}{2} < 1 \) so the series converges by the nth root test.
(9a). Conditionally convergent: The series converges by the alternating series test since
\[ \frac{10}{7n+2} > \frac{10}{7(n+1)+2} \] and \( \lim_{n \to \infty} \frac{10}{7n+2} = 0 \) but not absolutely since \( \sum_{n=1}^{\infty} \left| \frac{10}{7n+2} \right| = \sum_{n=1}^{\infty} \frac{10}{7n+2} \) diverges by comparing it with the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \), which diverges, using the limit comparison test:
\[ \lim_{n \to \infty} \frac{\frac{10}{7n+2}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{10n}{7n+2} = \frac{10}{7} < \infty. \]

(9b). Absolutely convergent: \( \sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{\sqrt{n}^5} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \) a convergent p-series with \( p = 5/2 > 1. \)

(9c). Absolutely convergent: \( \sum_{n=0}^{\infty} |(-1)^n 5^{-n}| = \sum_{n=0}^{\infty} 5^{-n} \) is a convergent geometric series with common ratio \( r = 1/5 < 1. \)

(9d). \( \lim_{n \to \infty} \frac{n^2 - n - 1}{2n^2 + n + 1} = \lim_{n \to \infty} \frac{\frac{1-n}{1-n} \frac{1}{n^2}}{\frac{2}{1-n} + \frac{1}{n^2}} = \frac{1}{2} > 0 \) so the series diverges by the nth term test for divergence.

(10a). The power series converges when \(|x - 1| < 1\) by the ratio test, which gives a radius of convergence 1 and interval of convergence centered at 1. The series diverges at \( x = 2 \) (harmonic series) but converges at \( x = 0 \) (alternate harmonic series), so the interval of convergence is \( 0 \leq x < 2. \)

(10b). The power series converges when \(|x - 1| < 1\) by the ratio test, which gives a radius of convergence 1 and interval of convergence centered at 1. The series diverges at \( x = 0 \) (harmonic series) but converges at \( x = 2 \) (alternate harmonic series), so the interval of convergence is \( 0 < x \leq 2. \)

(10c). The power series converges when \(|x + 1| < 5\) by the ratio test, which gives a radius of convergence 5 and interval of convergence centered at \(-1\). The series diverges at \( x = 4 \) (harmonic series) but converges at \( x = -6 \) (alternate harmonic series), so the interval of convergence is \(-6 \leq x < 4. \)

(10d). The power series converges when \(|x + 1| < 5\) by the ratio test, which gives a radius of convergence 5 and interval of convergence centered at \(-1\). The series diverges at \( x = -6 \) (harmonic series) but converges at \( x = 4 \) (alternate harmonic series), so the interval of convergence is \(-6 < x \leq 4. \)

(11a). \( p_2(x) = e^2 - 2e^2(x + 1) + 2e^2(x + 1)^2 \)

(11b). \( p_2(x) = 1 - \frac{25}{2} (x - 2\pi)^2 \)

(12a). \( p_3(x) = 1 + \frac{e}{3} - \frac{e}{2} x - \frac{e}{6} x^3 \)

(12b). \( p_3(x) = 1 - \frac{1}{2} (x - \frac{\pi}{2})^2 \)