MAT 1575 Final Exam Review Problems

Revised by Prof. Kostadinov Spring 2014, Prof. ElHitti Summer 2017, Prof. Africk Fall 2019

1. Evaluate the following definite integrals:
   a. \[ \int_{0}^{1} x^2 (x^3 + 1)^3 \, dx \]
   b. \[ \int_{0}^{1} \frac{x}{\sqrt{x^2 + 9}} \, dx \]
   c. \[ \int_{0}^{1} \frac{3x^2}{\sqrt{x^3 + 1}} \, dx \]

2. Evaluate the following indefinite integrals:
   a. \[ \int x^2 \ln(x) \, dx \]
   b. \[ \int x^2 e^{-x} \, dx \]
   c. \[ \int x \cos(3x) \, dx \]

3. Find the area of the region enclosed by the graphs of:
   a. \[ y = 3 - x^2 \quad \text{and} \quad y = -2x \]
   b. \[ y = x^2 - 2x \quad \text{and} \quad y = x + 4 \]

4. Find the volume of the solid obtained by rotating the region bounded by the graphs of:
   a. \[ y = x^2 - 9, \ y = 0 \quad \text{about the x-axis.} \]
   b. \[ y = 16 - x, \ y = 3x + 12, \ x = 1 \quad \text{about the x-axis.} \]
   c. \[ y = x^2 + 2, \ y = -x^2 + 10, \ x \geq 0 \quad \text{about the y-axis.} \]

5. Evaluate the following indefinite integrals:
   a. \[ \int \frac{1}{x^2 \sqrt{36-x^2}} \, dx \]
   b. \[ \int \frac{\sqrt{x^2-9}}{x^4} \, dx \]
   c. \[ \int \frac{9}{x^2 \sqrt{x^2+9}} \, dx \]
   d. \[ \int \frac{6}{x^2 \sqrt{x^2-36}} \, dx \]

6. Evaluate the following indefinite integrals:
   a. \[ \int \frac{3x+7}{x^2+6x+9} \, dx \]
   b. \[ \int \frac{5x+6}{x^2-36} \, dx \]
   c. \[ \int \frac{3x+2}{x^2+2x-8} \, dx \]
   d. \[ \int \frac{12-8x}{x^2(x-6)} \, dx \]
   e. \[ \int \frac{-2x^2+4x+4}{x(x-2)^2} \, dx \]

7. Evaluate the improper integral:
   a. \[ \int_{0}^{\infty} \frac{2}{(x+2)^3} \, dx \]
   b. \[ \int_{0}^{\infty} \frac{5}{\sqrt{x}+5} \, dx \]
   c. \[ \int_{1}^{\infty} \frac{3}{\sqrt[3]{(x-3)^4}} \, dx \]

8. Decide if the following series converges or not. Justify your answer using an appropriate test:
   a. \[ \sum_{n=1}^{\infty} \frac{9n^5}{3n^5+5} \]
   b. \[ \sum_{n=1}^{\infty} \frac{5}{10^n} \]
   c. \[ \sum_{n=1}^{\infty} \frac{5n}{10^n} \]
   d. \[ \sum_{n=1}^{\infty} \frac{n!}{n^2 5^n} \]
   e. \[ \sum_{n=1}^{\infty} \left( \frac{n+1}{2n+3} \right)^n \]

9. Determine whether the series is absolutely or conditionally convergent or divergent:
   a. \[ \sum_{n=1}^{\infty} \frac{(-1)^n 10}{7n+2} \]
   b. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^5}} \]
   c. \[ \sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{(2n+1)} \]
   d. \[ \sum_{n=1}^{\infty} \frac{n^2 - n - 1}{2n^2 + n + 1} \]
10. Find the radius and the interval of convergence of the following power series:

\[ \sum_{n=0}^{\infty} \frac{(x-1)^n}{n+2} \]
\[ \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^n}{n+2} \]
\[ \sum_{n=1}^{\infty} \frac{(x+1)^n}{n5^n} \]
\[ \sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n5^n} \]

11. Find the Taylor polynomial of degree 2 for the given function, centered at the given number a:

a. \( f(x) = e^{-2x} \) at \( a = -1 \)

b. \( f(x) = \cos(5x) \) at \( a = 2\pi \)

12. Find the Taylor polynomial of degree 3 for the given function, centered at the given number a:

a. \( f(x) = 1 + e^{-x} \) at \( a = -1 \)

b. \( f(x) = \sin(x) \) at \( a = \frac{\pi}{2} \)

Answers:

(1a). \( \frac{5}{4} \)  (1b). \( \sqrt{10} - 3 \)  (1c). \( \frac{3}{2} (2^{\frac{2}{3}} - 1) \)

(2a). \( \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + C \)  (2b). \( -(x^2 + 2x + 2)e^{-x} + C \)  (2c). \( \frac{1}{3}x\sin(3x) + \frac{1}{9}\cos(3x) + C \)

(3a). The area of the region between the two curves is:

\[ \text{Area} = \int_{-1}^{3} \left( 3 - x^2 - (-2x) \right) dx = \frac{32}{3} \]

(3b). The area of the region between the two curves is:

\[ \text{Area} = \int_{-1}^{4} \left( x + 4 - (x^2 - 2x) \right) dx = \frac{125}{6} \]
(4a). Approximate the volume of the solid by vertical disks with radius \( y = x^2 - 9 \) between \( x = -3 \) and \( x = 3 \); gives the volume is 
\[
V = \int_{-3}^{3} \pi (x^2 - 9)^2 \, dx = \frac{1296}{5} \pi.
\]

(4b). Using a washer of outer radius \( R_{outer} = 16 - x \) and inner radius \( R_{inner} = 3x + 12 \) at \( x \), gives the volume:
\[
V = \pi \int_{-1}^{1} [(16 - x)^2 - (3x + 12)^2] \, dx = \frac{656\pi}{3},
\]
where the upper limit 1 is obtained from \( 16 - x = 3x + 12 \Rightarrow x = 1 \).

(4c). \( 16\pi \)

(5a). \( \frac{\sqrt{36 - x^2}}{36} + C \)  
(5b). \( \frac{(x^2 - 9)^{3/2}}{27x^3} + C \)  
(5c). \( -\frac{\sqrt{x^2 + 9}}{x} + C \)  
(5d). \( \frac{\sqrt{x^2 - 36}}{6x} + C \)

(6a). \( \frac{2}{x+3} + 3\ln|x + 3| + C \)  
(6b). \( 3\ln|x - 6| + 2\ln x + 6 + C \)  
(6c). \( \frac{5}{3}\ln|x + 4| + \frac{4}{3}\ln|x - 2| + C \)

(6d). \( \frac{2}{x} - \ln|x - 6| + \ln|x| + C \)  
(6e). \( \ln|x| - \frac{2}{x-2} - 3\ln|x - 2| + C \)

(7a). \( \frac{1}{4} \)  
(7b). The integral does not converge  
(7c). The integral does not converge

(8a). \( \lim_{n \to \infty} \frac{9n^5}{3n^5 + 5} = \lim_{n \to \infty} \frac{3}{1 + 5/3n^5} = 3 > 0 \), so the series diverges by the nth term test for divergence.

(8b). This is a geometric series, with common ratio \( r = 1/10 < 1 \), so it converges to \( 5/9 \):
\[
\sum_{n=1}^{\infty} \frac{5}{10^n} = \frac{a}{1-r} = \frac{5/10}{9/10} = \frac{5}{9}.
\]

(8c). \( \lim_{n \to \infty} \frac{5(n+1)!}{10^{n+1}} / \frac{5n!}{10^n} = \lim_{n \to \infty} \frac{1 + 1/5n}{10} = \frac{1}{10} < 1 \), so the series converges by the ratio test.

(8d). \( \lim_{n \to \infty} \frac{(n+1)!}{(n+1)^25^{n+1}} / \frac{n!}{n^{2}5^n} = \lim_{n \to \infty} \frac{n^2}{5(n+1)} = \lim_{n \to \infty} \frac{n}{5(1+\frac{1}{n})} = \lim_{n \to \infty} \frac{n}{5} = \infty \), so the series diverges by the ratio test.

(8e). \( \lim_{n \to \infty} \left[ \left( \frac{n+1}{2n+3} \right)^n \right] = \lim_{n \to \infty} \frac{n+1}{2n+3} = \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{2 + \frac{3}{n}} = \frac{1}{2} < 1 \), so the series converges by the nth root test.
(9a). Conditionally convergent: The series converges by the alternating series test since
\[
\frac{10}{7n+2} > \frac{10}{7(n+1)+2} \quad \text{and} \quad \lim_{n \to \infty} \frac{10}{7n+2} = 0
\]
but not absolutely since \( \sum_{n=1}^{\infty} \left| (-1)^n \frac{10}{7n+2} \right| = \sum_{n=1}^{\infty} \frac{10}{7n+2} \)
diverges by comparing it with \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) which diverges, using the limit comparison test:
\[
\lim_{n \to \infty} \frac{\frac{10}{7n+2}}{\frac{1}{\sqrt{n}}} = \lim_{n \to \infty} \frac{10}{7n+2} \sqrt{n} = \infty.
\]

(9b). Absolutely convergent:
\[
\sum \text{ convergent } p\text{-series with } p = \frac{5}{2} > 1.
\]

(9c). Absolutely convergent:
\[
\sum (-1)^n \frac{5^{-n}}{n} \quad \text{is a convergent geometric series with}
\]
common ratio \( r = \frac{1}{5} < 1. \)

(9d). \[
\lim_{n \to \infty} \frac{n^2 - n - 1}{2n^2 + n + 1} = \lim_{n \to \infty} \frac{1 - \frac{n}{n^2}}{\frac{1}{2n^2} + \frac{1}{n} + \frac{1}{n^2}} = \frac{1}{2} > 0
\]
so the series diverges by the nth term test for divergence.

(10a). The power series converges when \(|x - 1| < 1\) by the ratio test, which gives a radius of convergence 1 and interval of convergence centered at 1. The series diverges at \( x = 2 \) (harmonic series) but converges at \( x = 0 \) (alternate harmonic series), so the interval of convergence is \( 0 \leq x < 2. \)

(10b). The power series converges when \(|x - 1| < 1\) by the ratio test, which gives a radius of convergence 1 and interval of convergence centered at 1. The series diverges at \( x = 0 \) (harmonic series) but converges at \( x = 2 \) (alternate harmonic series), so the interval of convergence is \( 0 < x \leq 2. \)

(10c). The power series converges when \(|x - 1| < 5\) by the ratio test, which gives a radius of convergence 5 and interval of convergence centered at \( -1. \) The series diverges at \( x = 4 \) (harmonic series) but converges at \( x = -6 \) (alternate harmonic series), so the interval of convergence is \( -6 \leq x < 4. \)

(10d). The power series converges when \(|x - 1| < 5\) by the ratio test, which gives a radius of convergence 5 and interval of convergence centered at \( -1. \) The series diverges at \( x = 4 \) (harmonic series) but converges at \( x = 4 \) (alternate harmonic series), so the interval of convergence is \( -6 < x \leq 4. \)

(11a). \[
p_2(x) = e^2 - 2e^2(x + 1) + 2e^2(x + 1)^2
\]

(11b). \[
p_2(x) = 1 - \frac{25}{2} (x - 2\pi)^2
\]

(12a). \[
p_3(x) = 1 + e - e(x + 1) + \frac{e}{2} (x + 1)^2 - \frac{e}{6} (x + 1)^3
\]
\[
= 1 + \frac{e}{3} - \frac{e}{2} x - \frac{e}{6} x^3
\]

(12b). \[
p_3(x) = 1 - \frac{1}{2} (x - \pi)^2
\]