SYSTEMS OF LINEAR EQUATIONS IN TWO VARIABLES (MAT 1190 SUPPLEMENTARY MATERIAL)

1. Introduction

We motivate this section with an example. Suppose that Adam has 7 bills, all fives and tens, and that the total value is $\$40$. How many of each bill does he have? We first define variables:

Let x be the number of five dollar bills.
Let y be the number of ten dollar bills.

Next, we write equations that describe the situation:

$$\begin{array}{cccccl} x & + & y & = & 7 & \text{: Adam has 7 bills.} \\ 5x & + & 10y & = & 40 & \text{: The combined value of the bills is \$40.} \end{array}$$

That is, we must solve the following system of two linear equations in two unknowns:

$$\left(\begin{array}{ccccc} x & + & y & = & 7 \\ 5x & + & 10y & = & 40 \end{array}\right)$$

This chapter deals with solving systems of two linear equations in two unknowns, such as the one above. We will consider two different algebraic methods: The substitution method, and the elimination method.

2. The Substitution Method

In this section we consider solving systems of two linear equations in two unknowns using **the substitution method**. To illustrate, we solve the system discovered in the Introduction with the substitution method:

$$\left(\begin{array}{ccccc} x & + & y & = & 7 \\ 5x & + & 10y & = & 40 \end{array}\right) \tag{1}$$

To begin, we solve the first equation for one variable in terms of the other. In this case we will solve for y in terms of x:

$$\begin{aligned} & x + y = 7 \\ \Longrightarrow\quad & y = 7 - x \end{aligned}$$

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We now **substitute** $y = 7 - x$ into the second equation $5x + 10y = 40$:

$$5x \quad + \quad 10(7 - x) \quad = \quad 40$$

We now solve the above equation for x:

$$\begin{array}{rcll} 5x + 10(7 - x) & = & 40 & \\ 5x + 70 - 10x & = & 40 & \text{Distribute } 10 \text{ into the parenthesis.} \\ -5x + 70 & = & 40 & \text{Collect like terms.} \\ -5x & = & -30 & \text{Subtract } 70 \text{ from both sides.} \\ x & = & 6 & \text{Divide both sides by } -5. \end{array}$$

Hence, we have that $x = 6$. To find y, we substitute $x = 6$ into the first equation of (1) and solve for y (Note: We may substitute into either equation or the equation $y = 7 - x$):

$$\begin{array}{ll} 6 + y = 7 & \\ y = 1 & \text{Subtract } 6 \text{ from both sides.} \end{array}$$

Hence, the solution to the system of linear equation (1) is

$$x = 6, \quad y = 1.$$

Before we are truly finished, we should **check** our solution. The solution of the system of equations are the values of the variables that make both equations correct at the same time. So to check we substitute $x = 6, y = 1$ into the original system of equations (1):

$$x + y = 7 \Longrightarrow 6 + 1 = 7 \Longrightarrow 7 = 7 \text{ correct!}$$
$$5x + 10y = 40 \Longrightarrow 5(6) + 10(1) = 40 \Longrightarrow 30 + 10 = 40 \Longrightarrow 40 = 40 \Longrightarrow \text{ correct!}$$

Hence, our answer is correct. To answer the original word problem–recalling that x is the number of five dollar bills and y be the number of ten dollar bills–we have that:

Adam has 6 *five dollar bills and* 1 *ten dollar bill.*

To summarize the steps we followed to solve a system of linear equations in two unknowns using the algebraic method of substitution:

1. **Solve one equation for one variable.**
2. **Substitute the quantity found in step 1 into the other equation.**
3. **Solve the resulting equation.**
4. **Substitute the value from step 3 back into the equation in step** 1 **to find the value of the remaining variable.**
5. **Check your solution!**
6. **Answer the question if it is a word problem.**

A system of two linear equations in two unknowns may have one solution, no solutions, and infinitely many solutions. In the example below we will see a system with no solutions.

Example 3.1: No Solution. Solve the following system of equations by substitution.

$$\left(\begin{array}{c} x + y = 1 \\ y = -x + 2 \end{array}\right)$$

The second equation is already solved for y in terms of x. Substitute the second equation into $x + y = 1$:

$$x + (-x + 2) = 1 \Longrightarrow 2 = 1 \text{ A contradiction!}$$

Since we get the contradiction $2 = 1$, there is no solution. □

Exercises

Solve the following systems of linear equations in two unknowns using the substitution method.

a. $\left(\begin{array}{ccccc} x & + & y & = & 3 \\ x & + & 2y & = & 5 \end{array}\right)$

b. $\left(\begin{array}{ccccc} -x & + & 2y & = & 1 \\ x & + & y & = & 2 \end{array}\right)$

c. $\left(\begin{array}{ccccc} x & - & y & = & 1 \\ x & + & 2y & = & 4 \end{array}\right)$

d. $\left(\begin{array}{ccccc} 2x & + & y & = & 1 \\ x & + & 2y & = & 2 \end{array}\right)$

e. $\left(\begin{array}{ccccc} 2x & - & y & = & 1 \\ 2x & - & y & = & 2 \end{array}\right)$

f. $\left(\begin{array}{ccccc} 2x & - & y & = & 1 \\ -x & + & 2y & = & 2 \end{array}\right)$

Solutions

a. $x = 1, y = 2$
b. $x = 1, y = 1$
c. $x = 2, y = 1$
d. $x = 0, y = 1$
e. No solution
f. $x = \frac{4}{3}, y = \frac{5}{3}$

3. The Elimination Method

A second algebraic method for solving a system of linear equations is **the elimination method**. The basic idea of the method is to get the coefficients of one of the variables in the two equations to be additive inverses, such as -3 and 3, so that after the two equations are added, one of the variables is eliminated. Let's use the same example in the Introduction Section to illustrate the method:

$$\begin{pmatrix} x & + & y & = & 7 \\ 5x & + & 10y & = & 40 \end{pmatrix}$$

The coefficients of x variable in in two equations are 1 and 5, respectively. We can make the coefficients of x to be additive inverses by multiplying the first equation by -5 and keeping the second equation untouched:

$$\begin{pmatrix} x & + & y & = & 7 \\ 5x & + & 10y & = & 40 \end{pmatrix} \Longrightarrow \begin{pmatrix} (-5)(x & + & y) & = & (-5)7 \\ 5x & + & 10y & = & 40 \end{pmatrix}$$

Use the distributive property, we rewrite the first equation as:

$$-5x \quad - \quad 5y \quad = \quad -35$$

Now we are ready to add the two equations to eliminate variable x and solve the resulting equation for y:

$$\begin{array}{rrcrcl} & -5x & - & 5y & = & -35 \\ + & 5x & + & 10y & = & 40 \\ \hline & & & 5y & = & 5 \\ & & \Longrightarrow & y & = & 1 \end{array}$$

To find x, we can substitute $y = 1$ into either equation of the original system to solve for x:

$$x + 1 = 7 \quad \Longrightarrow \quad x = 6$$

Hence, we get the same solution as we obtained using the substitution method in the previous section:

$$x = 6, \quad y = 1.$$

In this example, we only need to multiply the first equation by a number to make the coefficients of variable x additive inverses. Sometimes, we need to multiply both equations by two different numbers to make the coefficients of one of the variables additive inverses. To illustrate this, let's look at the following system of equations:

$$\begin{pmatrix} -3x & + & 2y & = & 3 \\ 4x & - & 3y & = & -6 \end{pmatrix} \tag{2}$$

Lets aim to eliminate the y variable here. Since the least common multiple of 2 and 3 is 6, we can multiply the first equation by 3 and the second equation by 2, so that the coefficients of y are additive inverses:

$$\left(\begin{array}{cccc} -3x & + & 2y & = & 3 \\ 4x & - & 3y & = & -6 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc} (3)(-3x & + & 2y) & = & (3)3 \\ (2)(4x & - & 3y) & = & (2)(-6) \end{array}\right)$$

Using the distributive property, we rewrite the two equations as:

$$\left(\begin{array}{ccccc} -9x & + & 6y & = & 9 \\ 8x & - & 6y & = & -12 \end{array}\right)$$

Adding them together gives:

$$-1x = -3 \quad \Longrightarrow \quad x = 3$$

To find y, we can substitute $x = 3$ into the first equation (or the second equation) of the original system to solve for y:

$$-3(3) + 2y = 3 \implies -9 + 2y = 3 \implies 2y = -12 \implies y = -6$$

Hence, the answer to the problem is:

$$x = 3 \quad y = -6$$

We can check the answer by substituting both numbers into the original system and see if both equations are correct.

The following steps summarize how to solve a system of equations by the elimination method:

1. **Multiply one or both equations by a nonzero number so that the coefficients of one of the variables are additive inverses.**
2. **Add equations to eliminate the variable.**
3. **Solve the resulting equation.**
4. **Substitute the value from step 3 back into either of the original equations to find the value of the remaining variable.**
5. **Check your solution!**
6. **Answer the question if it is a word problem.**

EXERCISES

Solve the following systems of linear equations in two unknowns using the elimination method.

a. $\left(\begin{array}{ccccc} x & + & 2y & = & -5 \\ 3x & + & 3y & = & 9 \end{array}\right)$

b. $\left(\begin{array}{ccccc} 2x & + & 2y & = & 2 \\ 4x & - & 3y & = & 18 \end{array}\right)$

c. $\left(\begin{array}{ccccc} -3x & - & 4y & = & 6 \\ 2x & + & 3y & = & -3 \end{array}\right)$

d. $\begin{pmatrix} 2x & + & y & = & 1 \\ x & + & 2y & = & 2 \end{pmatrix}$

e. $\begin{pmatrix} 2x & - & y & = & 1 \\ -x & + & 2y & = & 2 \end{pmatrix}$

Solutions

a. $x = 11, y = -8$
b. $x = 3, y = -2$
c. $x = -6, y = 3$
d. $x = 0, y = 1$
e. $x = \frac{4}{3}, y = \frac{5}{3}$

4. Word Problems

In this section, you will consider various word problems that require you to solve a system of two equations in two unknowns. You may use either the substitutions method, or the elimination method as you deem appropriate. In particular, you will:

1. Define the variables that you want to find with "let" statements.
2. Create equations that express the information given in the problem.
3. Solve the system of equations by substitution or elimination.

Example. Denice bought 6 new compact disks and 2 used compact disks for $\$72$. At the same prices, Jose bought 3 new compact disks and 8 used compact disks for $\$78$. Find the cost of buying a single new compact disk.

Solution. We first define the variables with "let" statements:

Let x be the price of a new compact discs.
Let y be the price of a used compact discs.

Create equations that express the information given in the problem:

$$\begin{array}{ccccl} 6x & + & 2y & = & 72 & \text{: Denice's compact disc purchases.} \\ 3x & + & 8y & = & 78 & \text{: Jose's compact disc purchases.} \end{array}$$

We solve the system of equations using substitution: First, solve the above equation $6x + 2y = 72$ for y:

$$\begin{array}{rll} & 6x + 2y = 72 & \\ \Longrightarrow & 2y = -6x + 72 & \text{Subtract } 6x \text{ from both sides.} \\ \Longrightarrow & y = -3x + 36 & \text{Divide both sides by 2.} \end{array}$$

Substitute $y = -3x + 36$ into the second equation $3x + 8y = 78$:

$$\begin{aligned} & 3x + 8y = 78 \\ \implies & 3x + 8(-3x + 36) = 78 \\ \implies & x = 10 \end{aligned}$$

Hence $x = 10$, thus the price of a new compact disc is $\$10$.

We could also solve for y, but it is not necessary. The question only asks us to find $x =$ the price of a new compact disc. □

EXERCISES

Solve the following word problems.

a. Galina spent $\$1.60$ for stamps to mail packages. Some were 30 cent stamps and the rest were 20 cent stamps. The number of 20 cent stamps was 2 less than the number of 30 cent stamps. How many stamps of each kind did Galina buy?
b. In yesterdays basketball game, Cheryl scored 17 points with a combination of 2-point and 3-point baskets. The number of 2-point shots she made was one greater than the number of 3-point shots she made. How many of each type of basket did she score?
c. A catering company is setting up tables for a big event that will host 764 people. When they set up the tables they need 2 forks for each child and 5 forks for each adult. If the company ordered a total of 2992 forks, how many adults and how many children will be attending the event?

SOLUTIONS

a. Four 30 cent stamps, and two 20 cent stamps.
b. Four 2 point shots, and three 3 point shots.
c. 276 children, 488 adults.

REFLECTION QUESTIONS

a. What are the advantages and disadvantages of solving a system of linear equation in two unknowns by the substitution method?

b. What are the advantages and disadvantages of solving a system of linear equation in two unknowns by the elimination method?

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