MAT 1275 Workbook

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Contents

Introduction

The problem-solving approach we use in this manuscript is based on the 6-point process developed by Professors Rojas and Benakli [1].

1. **Context**: What is the problem about?
2. **Observations**: List as many observations as possible (at least three). Include key words and symbols.
3. **Questions**: Write down (at least three) questions you can ask about the problem. Be sure to include any questions you have relating to the observations you have made.
4. **Strategies**: Write down the plan or action strategy.
5. **Concepts**: Write down concepts needed to understand and solve the problem.
6. **Conclusions**: Use complete sentences to express the conclusion.

For each homework set, an example is provided demonstrating this process which may not be the simplest of the problem types but gives a flavor of the process for the topic in general. The example does not mirror the solutions to the suggested problems but may inspire thought about the problems. The purpose is to help structure the thought process involved in solving problems. While the process isn't necessary to arrive at 'the answer', it is necessary to arrive at a flexibility of mind needed to use the concepts in unfamiliar situations.

The discussion problems, the (6-point) problems, and the exercises may be used to generate class discussion or to assign as homework or as individual or group class work. Even if the 6-point process isn't required writing, the thought process should be encouraged on all problems.

In addition, it is the authors' feeling that motivating the topics using modeling problems may be useful to generate interest and a clear idea of the purpose of variables. Understanding a modeling problem and the assumptions that may be needed can happen on day one (covering context and some observations or questions). You may find it helpful to suggest related/toy problems in stages according to the topic that is being discussed. Here is a selection which collectively touches on many of the topics in this workbook:

- Suppose you have access to a mixture of one part fertilizer to 5 parts water and also to another mixture of 2 parts fertilizer to 7 parts water. How can you make a mixture of 2 gallons of a mixture which is 3 parts fertilizer to 11 parts water?

[1] This is based on the chapter book Mathematical Literacy and Critical Thinking by Rojas, E., & Benakli, N. (2020). In: But, J. (eds) Teaching College-Level Disciplinary Literacy. Palgrave Macmillan, Cham.

- According to historical data [2] average temperatures for Central Park in January from 2010 until 2019 were 32.5, 29.7, 37.3, 28.6, 29.9, 34.5, 38, 31.7, 32.5, and 39.1 degrees Fahrenheit, respectively. Using this data, predict the average temperature for the years 2040 and 2041. How good was your prediction (check source for actual averages)?

- Suppose you have a paraboloid reflective surface that focuses the light rays at its focus. How wide should you make a grill so that the food sitting on it will be heated? (Hint: The focus of a parabola of the form $y = ax^2 + bx + c$ is on the axis of symmetry and $\frac{1}{4a}$ units away from the vertex ('inside' the parabola)).

- If you know you are 3 miles from cell tower A and 2 miles from cell tower B, where might you be? How many cell towers would you need to know your distance to to determine your location? Perhaps consider the two dimensional analogue.

- Suppose you have 1000 feet of fencing. How do you enclose the largest possible rectangular area?

- Suppose Holly has a credit card with an interest rate of 19 percent annually (compounded either monthly or continuously). If she makes a \$50 purchase and only pays 1 percent per month, how much will she owe after 3 years? 4 years? 5 years?

- Suppose a long hallway of width 5 feet meets a long hallway of width 4 feet at a right angle. Assume the ceiling height is 8 feet. What is the length of the longest pipe you can fit move through the L-shaped passage?

[2] https://www.weather.gov/media/okx/Climate/CentralPark/monthlyannualtemp.pdf

1 Order of Operations

1.1 Discussion Problems

1. Give an example of an expression where parentheses are not superfluous (that the value of the expression changes upon erasing the parentheses).
2. What is the geometric meaning of the absolute value? Give an example.
3. Why is the product of two negative numbers positive? (you could try to give a sensible word problem to explain this)
4. What does it mean to factor a number? Give an example listing all factors of a number (be sure to also include the negative ones).
5. What is the least common multiple of 100 and 150?
6. Why does adding fractions require a common denominator? (draw pictures)
7. Give an example of reducing a fraction and explain why the original and the reduced fractions have the same value by drawing a picture.
8. Draw a picture (calculating the area of a rectangle) that shows why

$$2(3+4) = 2 \cdot 3 + 2 \cdot 4.$$

1.2 Example of the 6 point process

Evaluate $3(2+3)(-2) - 2/3$.

- **Context**:

  Evaluation

  Order of operations

  Parentheses

  Fractions

- **Observations**:

  Parentheses appear.

  Multiplication, addition and subtraction appear.

  The expression contains a fraction.

  There are negative and positive numbers.

- **Questions**:

  What is the purpose of the parentheses? To prioritize what is contained between the parentheses if it differs from the order of operations. In this problem we should add the 2 and the 3 before doing any other calculation.

What does it mean if there is no sign between a number and a parenthesis? This means the number should be multiplied by the result of simplification of the number in parentheses.

What is the order of operations? From left to right, simplify what is in parentheses first, then multiplication and division, then addition and subtraction.

What operation corresponds to a fraction? 2/3 can be thought of as a fraction (two of the three parts making up one whole) or as the result of the division of 2 by 3.

How do we add/subtract fractions?

What is the form of the answer? It should be a number. It will be a negative number (why?) and it will be a fraction.

- **Strategies**:

  Proceed with the order of operations using equal signs where appropriate. When subtracting the fraction, we may need to find a least common denominator or since the fraction will be subtracted from an integer, we may be able to subtract by visualizing the subtraction.

  Note: It may be possible to do this in various ways, for example, if reordering terms is beneficial, or the arithmetic involving fractions may be done in different ways.

- **Concepts**: Write down concepts needed to understand and solve the problem.

  Arithmetic of positive and negative numbers.

  Addition/subtraction of fractions.

  The order of operations.

$$\begin{aligned}
3(2+3)(-2) - 2/3 &= 3 \cdot 5(-2) - \frac{2}{3} \\
&= 15 \cdot (-2) - \frac{2}{3} \\
&= -30 - \frac{2}{3} \\
&= -\frac{90}{3} - \frac{2}{3} \\
&= -\frac{92}{3} \\
&= -30\,\frac{2}{3}
\end{aligned}$$

Note: The multiplication $3 \cdot 5 \cdot (-2)$ can be done in any order, for example, $3 \cdot 5 \cdot (-2) = 3 \cdot (-10) = -30$. And the subtraction $-30 - \frac{2}{3}$ may be visualized and seen to be written as $-30\,\frac{2}{3}$ directly.

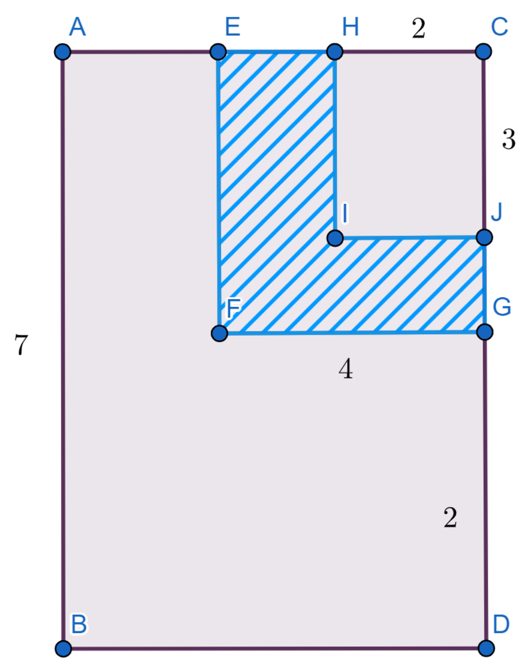
- **Conclusions**:

  The result of the calculation is $3(2+3)(-2) - 2/3 = -\frac{92}{3}$.

  Note: To find it on a number line, it may be more convenient to write it as a mixed number: $3(2+3)(-2) - 2/3 = -30\frac{2}{3}$.

1.3 Problems (6 pt Problems)

1. Evaluate $-2|3-7| + 2$.
2. Evaluate $\frac{-2+4}{-5(2)+7} - \frac{3}{5}$.
3. Suppose a carpet costs \$2 a square foot to purchase and install. How much would it cost to carpet the shaded region where $AB = 7$ ft, $HC = 2$ ft, $CJ = 3$ ft, $FG = 4$ ft, and $GD = 2$ ft? Assume all non-obtuse angles are right angles. (Be sure to mention assumptions related to the application in your 6 point discussion).



1.4 Exercises

1. Evaluate $2 \cdot 3 - 2(4 - 5)$.
2. Evaluate $-|2 - |3 - 6||$.
3. Evaluate $-(-2 + 2 \cdot 5) + 5 - 5 + 2(1 - 5)$.
4. Evaluate $4 \div 2 \cdot 2$.
5. Evaluate $\frac{2}{-5} - \frac{-2}{5}$.
6. Evaluate, reduce, and find on a numberline: $\frac{3}{10} - \frac{7}{6}$.
7. Evaluate, reduce, and find on a numberline: $\frac{3}{10} \cdot \frac{7}{6}$.
8. Estimate and verify your answer by direct computation: $\frac{-29}{5} \div 3 - \frac{4}{-3}$.
9. Evaluate and reduce $\frac{3}{2} \cdot \frac{2}{15} \cdot \frac{5}{4} \cdot \frac{11}{3} \cdot \frac{17}{19} \cdot \frac{19}{11}$.
10. Which fraction is bigger and why? $\frac{6}{15}$ or $\frac{7}{16}$?
11. Write $\frac{8}{3}$ as a mixed number and as a decimal.
12. Multiply $3\,\frac{1}{2}$ and $4\,\frac{1}{3}$.

2 Integer Exponents

2.1 Discussion Problems

1. In the expression 2^3, what is the base?
2. In the expression 2^3, what is the exponent? What does it count? In the expression $3 \cdot 2$ what does the 3 count?
3. In the expression $2^7 \cdot 2^8$, what does the 7 count? What does the 8 count? Write this expression in a more compact form.
4. Can you write $2^7 + 2^8$ in a more compact form? Explain.
5. In the expression $\frac{2^7}{2^8}$, what is the common factor (how many 2's cancel?)?
6. In the expression $(2^3)^4$, what is the 4 counting? And what is the 3 counting? Explain why this can be written as 2^{12}.
7. Explain why $(2^3 4^5)^4 = (2^3)^4(4^5)^4 = 2^{12}4^{20}$. How is this similar to the equality $4(3 \cdot 2 + 5 \cdot 4) = 12 \cdot 2 + 20 \cdot 4$?
8. Give an example of each of the 'rules' of exponents.
9. Explain why $7^0 = 1$.
10. Why are $(-3)^2$ and -3^2 different?
11. What is the meaning of a negative exponent? Give an example.
12. Using the meaning above, rewrite the following using positive exponent in the most compact form: $\left(\frac{1}{2}\right)^{-1}$ and $\left(\frac{2}{3}\right)^{-2}$.

2.2 Example of the 6 point process

Evaluate $\frac{3^5 2^3}{3^2 2^5}$.

- **Context**:

  Evaluation

  Fraction

  Multiplication

  Division

  Exponents

  Bases

- **Observations**:

  Some of the bases are the same.

  The fraction could possibly be reduced.

  There are two factors in the numerator and two factors in the denominator.

  There are no negative numbers.

- **Questions**:

  What is the role of a base and an exponent?

  What does the order of operations say about this problem?

  How do you reduce a fraction? You can divide numerator and denominator by a common factor.

  What is a common factor? Do the numerator and denominator have a common factor?

  What does it mean if there is no sign between 3^5 and 2^3, and between 3^2 and 2^5? It means these are multiplied.

  What is the form of the answer? It should be a number which may be a fraction. It may be convenient to keep exponents. If it is a fraction, it should be reduced.

- **Strategies**:

  Consider the meaning of the exponents and determine the common factors. Use this information to reduce the fraction.

  We could follow the order of operations and evaluate the numerator and denominator (multiply it all out). To reduce the fraction we need to identify common factors. But this sounds hard.

- **Concepts**:

  Write down the concepts needed to understand and solve the problem.

  Exponents

  Fraction reduction

  Consider $\frac{3^5 2^3}{3^2 2^5}$.

  The numerator has five 3's as factors and the denominator has two 3's as factors. So two of the 3's are common and cancel. We are left with three 3's in the numerator.

  The numerator has three 2's as factors and the denominator has five 2's as factors so three of them are common and cancel leaving two 2's in the denominator.

  Therefore, $\frac{3^5 2^3}{3^2 2^5} = \frac{3^3}{2^2}$.

- **Conclusions**:

$$\frac{3^5 2^3}{3^2 2^5} = \frac{3^3}{2^2} \quad \text{or} \quad \frac{3^5 2^3}{3^2 2^5} = \frac{27}{4} = 6\,\frac{3}{4}$$

2.3 Problems (6 pt Problems)

1. Write the following compactly using positive exponents: $5^{-2}5^2$.
2. Write the following compactly using positive exponents: $\left(\dfrac{5^{-2}3^2}{5^{-4}3^{-4}2^3}\right)^{-3}$.
3. If the sun is roughly 9.14×10^7 miles from the earth and light travels through space at roughly 6.71×10^8 miles per hour, roughly how many light years is the sun away from the earth? How many light minutes is this?

2.4 Exercises

1. Evaluate $(-2)^3$.
2. Evaluate $4^{-1} - 2^0 + 1$.
3. Evaluate $\left(\dfrac{2}{3}\right)^{-2} - 4^{-1}$.
4. Evaluate $\left(\dfrac{2^3 3^{-5} 5^{-17}}{2^{-2} 3^2 5^{-11}}\right)^{-3}$.
5. Estimate $(2.112 \times 10^3)(3.23 \times 10^{-7})$.
6. Which quantity is bigger and why? 5^{10} or 5^5.
7. Which quantity is bigger and why? 5^{-10} or 5^{-5}.

3 Polynomials: Evaluating, Adding, Subtracting, and Multiplying

3.1 Discussion Problems

1. What is a linear expression? Give an example.
2. What is the largest number of terms a polynomial of degree 3 with one variable can have?
3. What is the largest number of terms a polynomial of degree 3 with two variables can have?
4. What is the degree of the product of a polynomial of degree 2 and a polynomial of degree 1?
5. What is the degree of the sum of a polynomial of degree 2 and a polynomial of degree 1?
6. Consider the polynomial $x^2 + \frac{x}{3}$. What is the coefficient of x?
7. What is the leading coefficient of the product of a polynomial with leading coefficient 3 and a polynomial with leading coefficient -5?
8. What is the constant coefficient of the product of a polynomial with constant coefficient 3 and a polynomial with constant coefficient -5?
9. Consider the product of $3x^2 - 2x + 1$ and $-x^2 + 3x - 2$. What is the coefficient of the x^2 term? Explain without multiplying the two polynomials entirely.
10. Why is $-2x^2 + 3x - 1$ negative when evaluated at a negative number?
11. Why must you have like terms to combine?
12. Why does $(x + 2)(x + 3) = x^2 + 5x + 6$? Explain by interpreting as an area.
13. Explain how to construct Pascal's triangle and and how it relates to expressions like $(x + y)^5$.

3.2 Example of the 6 point process

Multiply and simplify $(2xy - 1)(x^2y + y + 1)$. Verify by evaluating at non-zero values for x and y.

- **Context**:

  Evaluation

  Multiplication

Exponents

Sums

Differences

Polynomials

Simplify

- **Observations**:

  There are two variables.

  It is a product of a binomial and a trinomial.

  The leading coefficients are 2 and 1.

  The constant coefficients are -1 and 1.

  There are negative coefficients.

  The two factors can't be simplified.

  When we evaluate $(2xy - 1)(x^2y + y + 1)$ and our answer we should get the same value.

- **Questions**:

  What does the order of operations say about this problem? It tells us how to evaluate the polynomial in various equivalent forms.

  How do you multiply polynomials? We distribute.

  How many terms should we expect after multiplying? We expect six terms.

  What do the leading coefficients of the factors tell us about our answer?

  What do the constant coefficients of the factors tell us about our answer?

  How do we add/subtract/multiply/divide integers?

  What does it mean to evaluate a polynomial? We replace each variable by a number and use the order of operations.

  What does it mean if there is no sign between $2xy - 1$ and $x^2y + y + 1$? That indicates multiplication.

  How do we simplify our answer?

  Identify like terms and 'count'.

  What is the form of the answer? It should be a polynomial with integer coefficients. It will have at most 6 terms. The leading coefficient will be 2 and the constant coefficient will be -1. We will also need to provide a value for x and a value for y to verify (this is not a proof, but support).

- **Strategies**:

  We will distribute to find the product which will give us 6 terms. We will then look for like terms and if we find any we will combine.

  We will use the order of operations to evaluate the original and our answer at $x = 2$ and $y = -3$

- **Concepts**:

  Write down concepts needed to understand and solve the problem.

  Polynomials

  Multiplication/Distribution

  Arithmetic of numbers

  Meaning of exponents

  Evaluation of polynomials

  The order of operations

  Equivalent expressions.

  Consider $(2xy-1)(x^2y+y+1)$. We distribute the $2xy$ and then the -1 into the second factor and then add:

  $$(2xy-1)(x^2y+y+1) = 2xyx^2y + 2xyy + 2xy - x^2y - y - 1 \cdot 1.$$

  Now we use the meaning of exponents and the fact that $1 \cdot 1 = 1$ to write the answer more compactly:

  $$(2xy-1)(x^2y+y+1) = 2x^3y^2 + 2xy^2 + 2xy - x^2y - y - 1.$$

  We note here that there are no like terms in the answer so this is our proposed simplified answer.

  We will replace x by 2 and y by -3 and evaluate both sides of the above equation to try to see if we have an error.

  The left side gives:

  $$\begin{aligned} & (2 \cdot 2(-3) - 1)(2^2(-3) + (-3) + 1) \\ = \; & (-12-1)(-12-3+1) \\ = \; & (-13)(-14) \\ = \; & 182. \end{aligned}$$

  The right side gives:

  $$\begin{aligned} & 2 \cdot 2^3(-3)^2 + 2 \cdot 2(-3)^2 + 2 \cdot 2(-3) - 2^2(-3) - (-3) - 1 \\ = \; & 2 \cdot 8 \cdot 9 + 2 \cdot 2 \cdot 9 - 12 - 4 \cdot (-3) + 3 - 1 \\ = \; & 144 + 36 - 12 + 12 + 3 - 1 \\ = \; & 182. \end{aligned}$$

- **Conclusions**:

  We found that

  $$(2xy-1)(x^2y+y+1) = 2x^3y^2 + 2xy^2 + 2xy - x^2y - y - 1.$$

  The left polynomial and the right polynomial are equivalent. We verified this for one particular evaluation and found there is no reason to suspect error.

3.3 Problems (6 pt Problems)

1. Write the following compactly (simplify): $(x^2 - 2x + 3) - (3x^2 - 2x - 1)$. Check your answer by evaluating at appropriate values.

2. Write the following compactly (simplify): $(x^2 - 2x + 3)(-2x - 1)$. Check your answer by evaluating at appropriate values.

3. Suppose you are interested in areas and perimeters of photographs of 3:2 aspect ratio. Write down a polynomial giving these areas. Draw a picture and label it. If you wanted a 1 inch border and thin frame, what is the area of the border and the length of frame material needed? Use reasonable approximations and indicate these in your discussion.

3.4 Exercises

1. Consider $-2x^3 + 3x^2 - 4x + 7$. Why is this a polynomial? What is its degree? What is its leading coefficient? How many terms does it have? What is the coefficient of x^2?

2. Consider $(3x - 2)(2x - 1)$. Why is this a polynomial? What is its degree? What is its leading coefficient? How many terms does it have? What is the coefficient of x?

3. Give an example of a product of two binomials.

4. Give an example of a product of a monomial with two variables and degree 3 and a binomial.

5. Simplify $2x^3 - 2x + 1 + (3x^3 - 2x^2 + 2x + 1)$.

6. Simplify $2x^3 - 2x + 1 - (3x^3 - 2x^2 + 2x + 1)$.

7. Multiply $(3x - 2)(3x + 2)$.

8. Consider $(-2x - 5)(-2x + 5)$. What is the coefficient of x? (try do find this without distributing completely)

9. Consider $(-2x + 3)(-2x + 5)$. What is the coefficient of x? (try do find this without distributing completely)

10. What is the degree and leading coefficient of $(2x-3)^7$. Find the coefficient of x^4 (Hint: use Pascal's triangle).

11. What is the degree and leading coefficient of $(3x-y)^7$. Find the coefficient of yx^4 (Hint: use Pascal's triangle).

4 Polynomial Division

4.1 Discussion Problems

1. How can we see division by a monomial as distribution of multiplication over addition? Give an example.

2. If you can divide a polynomial of degree 5 by a polynomial of degree 1, what degree is the quotient?

3. How are factoring and dividing related when dealing with numbers? And when dealing with polynomials? Give examples.

4. If you know that a certain polynomial can be divided by $x - 2$ with remainder zero, where can you evaluate that polynomial with the result of zero?

4.2 Example of the 6 point process

Divide $x^2 - 1$ by $x + 1$. Check for errors by evaluating appropriate expressions at a non-zero value.

- **Context**:

  Polynomials

  Division

- **Observations**:

  The dividend is of degree 2.

  The divisor is of degree 1.

  The coefficients are integers.

  Evaluating polynomials.

  Order of operations.

  We recall similar problems involving numbers.

- **Questions**:

  Is this similar to other problems we have seen? Division of numbers.

  How can we divide numbers? We can use long division and can remind myself with an example, or we can write as a fraction, factor the numerator to see how to reduce the fraction.

  What do the degrees of the dividend and the divisor tell us about our answer?

  What does the fact that the coefficients are integers tell us about our strategy or conclusion?

What form will our answer be? It will be a polynomial in x and of degree 1 if the remainder is zero.

How do I evaluate an expression?

How will I check my answer for errors? Either proceed with multiplication of the quotient by the divisor which should result in the dividend, or evaluate the fraction and the answer for one or more values to build confidence in the answer.

- **Strategies**:

  I can use long division of polynomials or I can factor (hoping that the remainder is zero) and reduce the fraction.

- **Concepts**:

  Multiplication and division of polynomials

  Polynomials

  If we divide $x^2 - 1$ by $x + 1$ and the remainder is zero, then we must be able to write $x^2 - 1 = (x + 1) \cdot (\text{a polynomial})$. This polynomial must be of degree one. In order for the two factors on the right give us the x^2 term on the left it must be:

  $$x^2 - 1 = (x + 1) \cdot (x + (\text{a number})).$$

  But in order to get the -1 on the left the number on the right must be so that $1 \cdot (\text{that number})$ is -1 so, it must be -1. Therefore,

  $$x^2 - 1 = (x + 1) \cdot (x - 1)$$

  if the remainder is zero. Distribution of the right hand side gives us $(x+1)(x-1) = x^2 + x - x - 1 = x^2 - 1$ so our problem can be written as

  $$\frac{x^2 - 1}{x + 1} = \frac{(x + 1)(x - 1)}{x + 1} = x - 1$$

  by noting the common factor $(x + 1)$.

- **Conclusions**:

  Dividing $x^2 - 1$ by $x + 1$ gives us $x - 1$. So,

  $$\frac{x^2 - 1}{x + 1} = x - 1,$$

  for all values of x where both sides make sense. Here, x can not have the value -1 because that would result in a division by zero on the left side of the above equation.

  To check for errors, we will evaluate at both sides of the above equation at 2:

  $$\frac{2^2 - 1}{2 + 1} = \frac{3}{3} = 1$$

and

$$2 - 1 = 1$$

so the two sides agree as they should.

4.3 Problems (6 pt Problems)

1. Divide $3x^2y^3 - 15xy^2 + 3xy$ by $3xy$ (check for errors by evaluation of appropriate expressions).

2. Divide $2x^4 - x^3 - 2x^2 + 5x - 2$ by $2x + 1$ (check for errors by evaluation of appropriate expressions).

3. Determine whether 5 is a zero of the polynomial

$$x^5 - 5x^4 - 2x^3 + 4x^2 + 29x + 4$$

by dividing by $x - 5$ and drawing appropriate conclusions.

4.4 Exercises

1. Divide $21x^5y^3$ by $7x^3y$ (check for errors by evaluation of appropriate expressions).

2. Divide $(x-2)(2x-3)$ by $x-2$ (check for errors by evaluation of appropriate expressions).

3. Divide $5x^2-7x-3$ by $3x+1$ (check for errors by evaluation of appropriate expressions).

4. Divide $-2x^3 + 3x^2 + x - 1$ by $-2x + 1$ (check for errors by evaluation of appropriate expressions).

5 Greatest Common Factor of Polynomials

5.1 Discussion Problems

1. What is the greatest common factor of two numbers? Give an example.
2. How can you tell when you have factored out the greatest common factor?
3. Give an example of a polynomial with greatest common factor $4xy^2z$.
4. If a polynomial has 4 terms and you factor out the greatest common factor, how many terms does the other factor have?
5. If you have a polynomial and can factor out x^2y, is $-x^2y$ also a divisor of the polynomial?
6. How are divisors and factors related?
7. Why might we be interested in factoring out the greatest common factor?

5.2 Example of the 6 point process

Factor out the greatest common factor of $3x^5 - 12x^4 + 9x^2$ (check for errors by evaluation of appropriate expressions).

- **Context**:

  Factoring

  Greatest common factor

  Polynomial

  Evaluating polynomials

- **Observations**:

  We are given a polynomial with one variable.

  The coefficients are all divisible by 3.

  The highest power of x is 5.

  The lowest power of x is 2.

  There are three terms in the given polynomial.

- **Questions**:

  How many variables should our answer have?

  What is a greatest common factor?

  How can we tell when we have factored out the greatest common factor?

  What does the fact that the given polynomial has three terms tell us about our answer? What is the form for our answer? Our answer should be a product of a monomial and a trinomial.

How will we check for errors?

We could distribute to see if we arrive at the given polynomial. We could also gain confidence by evaluating our answer and the original at one or more values.

- **Strategies**:

  Find the greatest common factor by guessing and checking by distributing.

- **Concepts**:

  Greatest common factor

  Polynomial

  Factor

  Multiplication

  Order of operations

  In considering $3x^5 - 12x^4 + 9x^2$, we see that we can divide each of the coefficients by 3 but no larger number and we can divide each term by x^2 but any larger power of x would result in a non-zero remainder. We therefore conclude that $3x^2$ is our greatest common factor. We can use division to find:

$$\frac{3x^5 - 12x^4 + 9x^2}{3x^2} = x^3 - 4x^2 + 3$$

  so,

$$3x^5 - 12x^4 + 9x^2 = 3x^2(x^3 - 4x^2 + 3).$$

- **Conclusions**:

$$3x^5 - 12x^4 + 9x^2 = 3x^2(x^3 - 4x^2 + 3).$$

  Since $(x^3 - 4x^2 + 3)$ has no common factor other than 1, we have factored the greatest common factor.

  Let us now check for errors by evaluating both sides of the above equation for some value of x, say, 2.

$$3 \cdot 2^5 - 12 \cdot 2^4 + 9 \cdot 2^2 = 96 - 192 + 36 = -60$$

  and

$$3 \cdot 2^2(2^3 - 4 \cdot 2^2 + 3) = 12(8 - 16 + 3) = 12(-5) = -60,$$

  which are equal as they should be, no matter the value that is substituted.

5.3 Problems (6 pt Problems)

1. Factor the GCF from $8x^2 - 10x^3 + 14x^5$ (check for errors by evaluation of appropriate expressions).

2. Factor the GCF from $-14x^5y^6 - 21x^4y^7$ so that the other factor has positive leading coefficient (check for errors by evaluation of appropriate expressions).

3. Factor the GCF from $(2x + 5) \cdot 7 - (2x + 5) \cdot 3x$ (check for errors by evaluation of appropriate expressions).

4. Factor $3x^2 + 6x - 5x - 10$ by first writing it as a sum of two binomials (pair the first two terms to form the first binomial and the last two terms to form the second binomial) and factor the greatest common factor from each binomial. Check for errors by evaluation of appropriate expressions.

5.4 Exercises

1. Factor out the GCF of $-5x^2 + 15x$.

2. Factor out the GCF of $16x^4y^6 - 20x^3y^7 + 6x^3y^9$.

3. Factor out the GCF of $-(x + 5)2x + 7(x + 5)$.

4. Factor $4x^2 - 2x + 6x - 3$ ‘by grouping’.

6 Factoring Polynomials

6.1 Discussion Problems

1. What does it mean to factor a polynomial?
2. What are the different strategies to factoring a quadratic polynomial? Give examples of each strategy. Are there examples where neither work? If so, provide such an example.
3. How is factoring related to division?
4. What does the sign of constant term of $ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$, tell you about its factors?
5. Is $(5x + 1)(30x + 7) = (30x + 7)(1 + 5x)$? Explain without distributing.

6.2 Example of the 6 point process

Factor $5x^2 - 29x - 6$.

- **Context**:

  Polynomial

  Factoring

- **Observations**:

  This is a quadratic polynomial.

  The greatest common factor is 1.

- **Questions**:

  What does it mean to factor?

  Is this problem similar to other problems we have seen? It is similar to factoring numbers.

  Where does the similarity end? The factoring process is different when there is a variable.

  What is the form of our answer? The product of two binomials if we succeed.

  What is the significance of the greatest common factor being 1?

  Can we check our answer? Yes, by distributing. We can also gain confidence by evaluating the original expression and our answer by evaluating at one or more different values.

- **Strategies**:

  Guess and check (possibly easy since 5 is prime).

  The '*ac* method'.

- **Concepts**:

  Factoring

  Distribution

  Polynomial

  To factor, we will first attempt the guess and check method (since it works, it is quick). We see the form is:

  $$5x^2 - 29x - 6 = (5x + a)(x + b),$$

  where a and b need to be determined. We know that if we multiply the right side, the constant term is ab which must agree with the constant term on the left. So ab must be -6. We have a few choices. But, to arrive at a number as large as 29, we probably should try to use the factors 6 and 1, one of which should be negative. But the coefficient of x is negative so it is also likely that -6 and 1 are the factors (the larger number is negative). By now, we are really hoping that this all works out!!

  Our proposed factoring is $(5x+1)(x-6)$, since to make a number as large as 29 the larger number 6 should be in the factor not including $5x$.

  Now we check the product:

  $$(5x + 1)(x - 6) = 5x^2 + x - 30x - 6 = 5x^2 - 29x - 6,$$

  which confirms that this is the correct factoring.

- **Conclusions**:

  We see that

  $$5x^2 - 29x - 6 = (5x + 1)(x - 6).$$

6.3 Problems (6 pt Problems)

1. Factor $72x^2y^3 - 18x^3y^2 + 15xy^5$. Check your answer by evaluating at an appropriate value.

2. Factor completely $36x^3 - 33x^2 + 6x$. Check by distributing your answer.

3. Factor completely $16x^4 - 1$ (over the integers). Check by distributing your answer.

4. If an object is $-16t^2 + 15t + 10$ feet from the ground when the clock reads t seconds, where is the object when the clock reads 1 second?

6.4 Exercises

1. Factor the GCF of $15x^5y^2 - 20x^3y^4z + 5x^2y^2$ out. Check your answer by distributing.

2. Factor $27x^4 - 18x^3 - 24x^2$ completely. Check your answer by distributing.

3. Factor $4y^2 - 9x^2$ completely. Check your answer by distributing.

7 Rational Expressions

7.1 Discussion Problems

1. What is a rational expression?
2. Give an example of a rational expression that is not a polynomial.
3. Demonstrate how we add/subtract/multiply/divide rational expressions by giving examples.
4. How could negative exponents be used to write rational expressions differently?
5. Can we evaluate a rational expression anywhere? If not, give an example.
6. Give an example of a reducible rational expression where the degree of the numerator is 3 and the degree of the denominator is 5. How do the degrees of the numerator and denominator relate to 3 and 5?
7. What is meant by a complex fraction?
8. Outline at least one strategy for simplifying a complex fraction. Give one example of your strategy where there are no variables, and one example where there is one variable.
9. If we are trying to simplify $\frac{(x-1)(x+2)}{2(x-3)} \cdot \frac{(x-1)(x-3)}{x(x+2)}$, what is the advantage to the numerators and denominators already being factored?
10. If we are trying to add $\frac{(x-1)}{2x(x-3)} + \frac{(x-1)}{x(x+2)}$ what is the advantage of finding the *least* common denominator?

7.2 Example of the 6 point process

Simplify

$$\frac{t + \frac{2}{t}}{t - \frac{1}{t}}.$$

Check for errors by evaluating this expression and your answer at some appropriate value.

- **Context**:

  Rational expression

  One variable

  Complex fraction

  Simplify

- **Observations**:

  The numerator has two terms, one of which is a fraction.

  The denominator has two terms, one of which is a fraction.

  This expression cannot be evaluated at 0.

- **Questions**:

  How do we add fractions?

  How do we divide fractions?

  What is the significance of the fact that this expression is undefined at 0?

  Is this problem (or parts of this problem) similar to another problem we have seen?

  What form will our answer take? Our answer should have at most one variable and if it is a fraction, the numerator and denominator should be polynomials.

  How do we know when our fraction is reduced completely? We will have to recognize that there are no common factors among the factors of our numerator and denominator.

  How can we check our answer? We can evaluate the original expression and our answer at a non-zero value (at least one) to gain confidence.

- **Strategies**:

  We will add fractions by finding the least common denominator and then divide. Finally we will reduce the fraction. We may need to factor a polynomial to reduce the fraction.

  Then we will check our answer by evaluating at 2, say.

- **Concepts**:

  Rational expression

  Adding and dividing rational expressions

  Reducing a rational expression

  Equivalent expressions

  By finding the least common denominator of the numerator and denominator, we can rewrite:

  $$\frac{t+\frac{2}{t}}{t-\frac{1}{t}} = \frac{\frac{t^2}{t}+\frac{2}{t}}{\frac{t^2}{t}-\frac{1}{t}} = \frac{\frac{t^2+2}{t}}{\frac{t^2-1}{t}},$$

  and by dividing fractions, we see that

  $$\frac{t+\frac{2}{t}}{t-\frac{1}{t}} = \frac{\frac{t^2}{t}+\frac{2}{t}}{\frac{t^2}{t}-\frac{1}{t}} = \frac{t^2+2}{t}\cdot\frac{t}{t^2-1} = \frac{(t^2+2)t}{(t^2-1)t}.$$

Now, we see that there is a common factor so the fraction can be reduced to give us

$$\frac{t+\frac{2}{t}}{t-\frac{1}{t}} = \frac{t^2+2}{t^2-1}.$$

Since the denominator can be factored as $(t+1)(t-1)$ and neither of these is a factor of the numerator, our rational expression can not be reduced.

- **Conclusions**:

  By adding, subtracting, dividing, and reducing, we found that

$$\frac{t+\frac{2}{t}}{t-\frac{1}{t}} = \frac{t^2+2}{t^2-1}$$

  for all values of t where both sides can be evaluated. We will check for errors by evaluating at 2 (we can't choose 0 or 1):

$$\frac{2+\frac{2}{2}}{2-\frac{1}{2}} = \frac{2+1}{\frac{3}{2}} = \frac{3}{1}\cdot\frac{2}{3} = 2$$

  and

$$\frac{2^2+2}{2^2-1} = \frac{6}{3} = 2.$$

  Since the two values agree, we gain confidence in our answer.

7.3 Problems (6 pt Problems)

1. Simplify $\left(\frac{2x^{-3}y^3z}{9x^{-5}yz^{-3}}\right)^{-3}$. Check your answer by evaluating at appropriate values.
2. Add and simplify $\frac{x-7}{x^2+x-2} + \frac{x+1}{x^2-x}$. Check your answer by evaluating at an appropriate value.
3. Simplify

$$\frac{\frac{2}{3(x-1)} - \frac{1}{3x}}{\frac{1}{x-1} - \frac{2}{3x}}.$$

   Check your answer by evaluating at an appropriate value.
4. If it costs 10 dollars for the loaf pans and two dollars a loaf for the rest of the materials, what is the average cost per loaf to make x loaves of bread? Test your answer in at least one particular case to see if it makes sense.

7.4 Exercises

1. Give an example of adding fractions and an example of multiplying fractions.

2. Simplify $\frac{6x^3y^3}{9x^5y}$. Check your answer by evaluating at appropriate values.

3. Simplify $\left(\frac{6x^3y^{-3}z^{-4}}{9x^{-5}yz^{-3}}\right)^{-2}$. Check your answer by evaluating at appropriate values.

4. Subtract and reduce: $\frac{7}{9x^2y} - \frac{3}{2xy^2}$. Check your answer by evaluating at an appropriate value.

5. Simplify $\frac{x-1}{(x+2)(x+1)} - \frac{2}{x(x+1)}$. Check your answer by evaluating at an appropriate value.

6. Simplify $\frac{x^2-x}{x^2-2x-3} \cdot \frac{x^2-2x}{4x^2-4}$. Check your answer by evaluating at an appropriate value.

7. Simplify $\frac{s-s^{-2}}{s^{-1}-s^2}$. Check your answer by evaluating at an appropriate value.

8. Simplify $\frac{\frac{2}{x-2}+\frac{1}{x+2}}{1-\frac{x}{2-x}}$. Check your answer by evaluating at an appropriate value.

9. Write an expression that represents the number of tasks per hour completed by a team consisting of Moira and Shelley if it takes Moira twice as long as it does Shelley to complete the task alone. Test your expression using a problem that you can figure out the answer without using your expression.

8 Radical Expressions

8.1 Discussion Problems

1. What does $\sqrt{2}$ mean?

2. Give the best integer (upper and lower) bounds for $\sqrt{5}$ without using a calculator. In other words, find the integer just smaller than $\sqrt{5}$ and the integer just greater than $\sqrt{5}$.

3. What is the process of simplifying the radical expression $\sqrt{50}$? Simplifying in this context means to write an equivalent expression with a number as small as possible under the radical.

4. How is the fact that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ for $a, b > 0$ related to the rules of exponents?

5. Explain why $\left(\sqrt[3]{x}\right)^3 = x$.

6. Give an example to show why $\sqrt{x^2 + 4} \neq x + 2$ even if $x > 0$.

7. In the expression $3\sqrt{5x} + 4\sqrt{5x}$, what are the 3 and 4 counting? How does this lead to writing the expression more compactly?

8. Explain why $(\sqrt{5} + \sqrt{3})^2 \neq 5 + 3$.

9. What does a rational exponent mean? Give an illustrative example.

8.2 Example of the 6 point process

Assume $x > 0$. Multiply and simplify $(\sqrt{x} - \sqrt{3})(\sqrt{x} + 2\sqrt{3})$.

- **Context**:

  Radical expressions

  One variable

  Multiplication

  Simplification

- **Observations**:

  There are parentheses.

  This is a product of two binomials.

  It is not a special product.

  There is one variable x which is assumed to be positive.

- **Questions**:

  What is the purpose of the parentheses?

  What is the operation between the two groupings? Multiplication.

  How many terms will we have after distributing?

  Is this similar to other problems we have seen? and if so, how is it different? Yes, for example, distribute $(x-3)(x+6)$. There are no radicals in this problem.

  How many variables will our answer contain? At most one, namely, x.

  Why is it important that x is positive? Because the square root would not give a real value if it were negative, it would be okay if $x = 0$.

  How do we multiply radicals?

  How do we simplify sums of radical expressions?

- **Strategies**:

  We will distribute to get 4 terms. We will then simplify each term and see if there are any like terms. If there are like terms, we will combine them. We will then check by estimating each side for some value of x that our answer holds up to scrutiny.

- **Concepts**:

  Radical expression

  Square root

  Multiplication/addition of square root

  Simplification of square root

  Following our strategy, we find

$$\begin{aligned}(\sqrt{x}-\sqrt{3})(\sqrt{x}+2\sqrt{3}) &= \sqrt{x}\sqrt{x}+2\sqrt{x}\sqrt{3}-\sqrt{3}\sqrt{x}-\sqrt{3}\cdot 2\sqrt{3}\\ &= x+2\sqrt{3x}-\sqrt{3x}-2\cdot 3\\ &= x+\sqrt{3x}-6.\end{aligned}$$

- **Conclusions**:

  We found that

$$(\sqrt{x}-\sqrt{3})(\sqrt{x}+2\sqrt{3}) = x+\sqrt{3x}-6$$

  for positive x.

  When x is replaced by 4, the left side is $(2-\sqrt{3})(2+2\sqrt{3})$. But $\sqrt{3}$ is between $\frac{3}{2}$ and 2 so this is a little smaller than $\frac{1}{2}\cdot 6 = 3$. The left side is $4+\sqrt{12}-6 = \sqrt{12}-2$ which is smaller than 2, but bigger then 1. This is consistent so we see no obvious sign of error.

  We could've checked the values when x is replaced with 3. The left side is 0. The right side is $3+\sqrt{9}-6 = 3+3-6 = 0$, which is also reassuring.

8.3 Problems (6 pt Problems)

1. Simplify $2\sqrt[3]{48x^9y^{18}z^5}$.

2. Multiply and simplify $(x-3-\sqrt{2})(x-3+\sqrt{2})$.

3. Simplify
$$\frac{1-\sqrt{2}}{1+\sqrt{3}} - \frac{2-\sqrt{2}}{1-2\sqrt{3}}$$
(write using at most three radicals).

4. Simplify and write your answer using radical notation (assume x and y are positive):
$$\sqrt{\frac{3\sqrt[3]{x^2y^2}}{4\sqrt[3]{3x^2\sqrt{3x}}}}.$$

8.4 Exercises

1. Evaluate exactly and estimate without a calculator at 4: $\sqrt[3]{10x^2}$.

2. Simplify $3\sqrt[4]{48x^9y^{18}z^6}$.

3. Evaluate $8^{2/3}$.

4. Simplify $\left(2\sqrt{\frac{2x^4y^3}{45z^7}}\right)^3$.

5. Simplify $5\sqrt{20}-3\sqrt{45}$.

6. Multiply and simplify $(7\sqrt{3}+2\sqrt{5})(2\sqrt{3}-3\sqrt{5})$.

7. Multiply and simplify $(\sqrt{3}+4\sqrt{5})(\sqrt{3}-4\sqrt{5})$.

8. Divide and simplify $\frac{7\sqrt{3}+2\sqrt{5}}{2\sqrt{3}-3\sqrt{5}}$.

9. Simplify and write your answer using radical notation (assume x and y are positive):
$$\left(\frac{3\sqrt{xy^3}}{4\sqrt[3]{(9x)^2\sqrt{3y}}}\right)^2.$$

9 Linear and Introduction to Quadratic Equations

9.1 Discussion Problems

1. How are expressions and equations different? Give examples.
2. What does it mean to be a solution to an equation? Give an example.
3. How is the order of operations when evaluating $3x + 2$ related to solving the equation $3x + 2 = 0$.
4. How can you recognize a linear equation? Give an example.
5. How can you recognize a quadratic equation? Give an example.
6. What is the zero product property? How is it used to solve a quadratic equation? Give an example.
7. How can solving a quadratic equation sometimes be reduced to solving two linear equations? Give an example.
8. What is involved in reducing the process of solving a quadratic equation to the process of solving two linear equations?
9. If $(x - 2)$ is a factor of a polynomial, why is 2 a zero?
10. If 3 is a zero of a polynomial, give a factor of that polynomial.
11. If we have an equation, and we subtract 2 from both sides of the equation, why is the resulting equation equivalent to the original? What does equivalence mean?
12. Given an equation, list 6 ways we can arrive at an equivalent equation.

9.2 Example of the 6 point process

Solve $2x^3 - 7x^2 = -6x$.

- **Context**:

  Polynomial equation

  Solution

- **Observations**:

  This is a polynomial equation. In fact, it is cubic since the highest power on x is 3 and no equivalent equation will reduce the highest power.

  Neither side of the equation is 0.

- **Questions**:

  How do we know this is an equation?

  How do we know it is a cubic equation?

  What are we looking for? What does it mean to be a solution to an equation?

  How do we solve a polynomial equation?

  What is standard form and why is it beneficial?

  How do we write an equivalent equation that is in standard form?

  What will our answer look like? It will be one, two, or three values that we can substitute in for x that will make the equation true.

  Does this look like other problems we have seen?

- **Strategies**:

  First we will write it in standard form by subtracting $6x$ from both sides.

  We will then try to factor the polynomial. This is then similar to other problems we have done.

  If we are able to factor it, then we will use the zero product property to find values that when substituted give a true statement.

- **Concepts**:

  Equivalent equations

  Polynomial equation

  Solution to an equation

  Factoring

  Zero product property

  We now consider $2x^3 - 7x^2 = -6x$. We are looking for values to substitute in for x that result in a true statement.

  We first add $6x$ to both sides to arrive at an equivalent equation:

  $$2x^3 - 7x^2 + 6x = 0,$$

  the left hand side of which we now attempt to factor. We first find the greatest common factor x and write the equivalent equation

  $$x(2x^2 - 7x + 6) = 0.$$

  Now we will factor the quadratic expression in the parentheses (using guess and check or the ac method): $2x^2 - 7x + 6 = (2x - 3)(x - 2)$ so that the given equation is equivalent to

  $$x(2x - 3)(x - 2) = 0.$$

So, we have a product of three linear expressions giving zero. When we evaluate this at a solution, we will get a product of three numbers resulting in zero which can only happen when one of the three numbers is zero (that is the zero product property). It follows that the solution we seek satisfies $x = 0, 2x - 3 = 0$, or $x - 2 = 0$. So, solving these linear equations either

$$x = 0$$

or (by undoing the multiplication by 2 and subtraction of 3)

$$x = \frac{3}{2}$$

or (by undoing the subtraction of 2)

$$x = 2.$$

- **Conclusions**:

  It follows that if x satisfies $2x^3 - 7x^2 = -6x$, then $x = 0, \frac{3}{2}$ or 2. Note that substituting each of these values into the equation $2x^3 - 7x^2 = -6x$ gives true statements.

9.3 Problems (6 pt Problems)

1. Solve $2(x - 3) - 1 = 4x + 2$.
2. Solve $6x(x + 1) = 4 - 4x$.
3. Suppose a rectangular piece of cardboard is 2 more feet long than it is wide and that the area is 15 square feet. Find the dimensions (the length and width) of the cardboard by drawing a picture, labeling the picture, writing down an appropriate equation and solving the equation. Be sure your answer makes sense in the application.

9.4 Exercises

1. Solve $4x - 2 = 0$.
2. Solve $2(x - 3) = 2x - 3(x - 1)$.
3. Solve $(x - 2)(7x - 3) = 0$.
4. Solve $10x^2 - 9x + 2 = 0$.
5. Solve $x(3x - 7) = -2$.
6. Suppose that you have a poster which is 2 feet by 3 feet. Suppose you can enlarge the poster by including a white border as long as its area doesn't exceed $8\,\frac{3}{4}$ square feet. What is the width of the largest uniform border you could have? Include a labeled picture and an appropriate equation.

7. Suppose an object has a height $-16t^2 + 78t + 10$ at when the stopwatch reads t seconds. At what time does it hit the ground? At what time is 10 feet above the ground?

10 Quadratic Equations

10.1 Discussion Problems

1. Give an example of a quadratic equation with no integer solutions.
2. What is the quadratic formula? Give an example of how to use it.
3. What is the process of completing the square? Give an example.
4. How many real solutions could a quadratic equation have? Give examples.

10.2 Example of the 6 point process

Suppose an object is $-16t^2 + 20t$ feet above the ground when the stopwatch reads t seconds. When will the object be 3 feet above the ground?

- **Context**:

  Application

  The time when the object is 3 feet above the ground.

  Quadratic expression

  Quadratic equation

- **Observations**:

  The answer should be positive.

  The expression giving the height is quadratic.

  The desired height is given as 3 feet.

  We are seeking the time.

- **Questions**:

  What do I know about quadratic expressions and equations?

  What equation might be helpful?

  What form will my answer take?

- **Strategies**:

  First find an equation so that the solution gives us the time that the object is 3 feet above the ground?

  Since the expression given is quadratic, this will likely lead to a quadratic equation which then we will solve by either factoring or using the quadratic formula.

  We will then see if the solutions makes sense.

- **Concepts**:

  Quadratic equation

  Quadratic formula

  Zero product property

  Solution

  Order of operations (for using quadratic formula)

  Since the expression $-16t^2 + 20t$ is the feet above ground and we are looking for the time when this is 3, we are looking for a solution to the equation

  $$-16t^2 + 20t = 3.$$

  We first write this in standard form by subtracting 3 from both sides of the equation to get $-16t^2 + 20t - 3 = 0$. A try at factoring is unsuccessful so we use the quadratic formula ($a = -16$, $b = 20$, $c = -3$):

  $$\begin{aligned} t &= \frac{-20 \pm \sqrt{20^2 - 4(-16)(-3)}}{2(-16)} \\ &= \frac{-20 \pm \sqrt{16(25-12)}}{-32} \\ &= \frac{-20 \pm 4\sqrt{13}}{-32} \\ &= \frac{5 \mp \sqrt{13}}{8}. \end{aligned}$$

  Note that $\sqrt{13}$ is between 3 and 4 so that $\frac{5-\sqrt{13}}{8}$ is positive, and, of course $\frac{5+\sqrt{13}}{8}$ is also positive.

- Conclusions:

  There are two times that the height is 3 feet: $\frac{5 \pm \sqrt{13}}{8}$.

  The smaller number is about $\frac{1}{8}$ seconds or a little bigger, and the other is about 1 or a little bigger. A calculator will help us with better approximations.

10.3 Problems (6 pt Problems)

1. Solve $(x-3)^2 = 5$.

2. Solve $x^2 - 6x - 2 = 0$ by completing the square.

3. Solve $3x(x-2) = 4$.

4. A triangle is 3 more feet high than its base. What are the base and the height of the triangle if its area is 10 square feet? Draw and label an appropriate figure and write down an appropriate equation as part of your solution.

10.4 Exercises

1. Solve $x^2 = 20$.
2. Solve $(x-2)^2 = 12$.
3. Solve $x^2 - 4x = 6$.
4. Solve $3x^2 - 4x = 20$ using the quadratic formula.
5. Suppose you are trying to make a square garden with a walkway of uniform width. You only have enough garden materials for a 10 foot by 10 foot gardening patch. How wide should your walkway be so that the total area (walkway and garden) is 120 square feet?
6. Suppose you want to form a box with an open top by cutting out corners of a rectangular piece of cardboard which is 10 inches by 7 inches. How high will the box be if the area of the base of the box is 50 square inches?
7. Suppose that a right triangle has a hypotenuse of length 5 inches and one of the legs is 2 inches more than the other. What are the lengths of the legs?

11 Complex Numbers and Quadratic Equations

11.1 Discussion Problems

1. What is the defining property of i?
2. How is the simplification of $3 + 2i - (5 - 7i)$ similar to the simplification of $3 + 2\sqrt{5} - (5 - 7\sqrt{5})$?
3. How is the simplification of $(3 + 2i)(5 - 7i)$ similar to the simplification of $(3 + 2\sqrt{5})(5 - 7\sqrt{5})$?
4. How is the simplification of $\dfrac{3 + 2i}{5 - 7i}$ similar to the simplification of $\dfrac{3 + 2\sqrt{5}}{5 - 7\sqrt{5}}$?
5. How many possible zeros does a quadratic expression have?
6. If a quadratic equation with integer coefficients has a solution of $4 - 2i$, what is the other solution?

11.2 Example of the 6 point process

Multiply $(3 - 2i)(-2 - 3i)$.

- **Context**:

  Complex numbers

  Arithmetic (multiplication/addition/subtraction)

- **Observations**:

  The numbers we are multiplying are complex numbers.

  This looks like other problems we have seen (for example, $(3 + 4\sqrt{2})(2 - 4\sqrt{2})$).

  This is a product of binomials.

- **Questions**:

  What is the defining property of i?

  How do we multiply/add/subtract complex numbers?

  How do we multiply binomials? We distribute.

  What form will our answer be? The standard form of a complex number is $a + bi$, where a and b are real numbers.

- **Strategies**:

  We will distribute as we have other binomials. Then we will replace i^2 with -1 and then collect like terms.

- **Concepts**:

  Complex numbers

  Binomials

  Arithmetic with complex numbers

  We first distribute to find

$$(3-2i)(-2-3i) = -6-9i+4i+6i^2.$$

  Since $i^2 = -1$ we see that

$$(3-2i)(-2-3i) = -6-9i+4i+6(-1) = -6-9i+4i-6.$$

  By collecting like terms, we find

$$(3-2i)(-2-3i) = -12-5i.$$

- **Conclusions**:

  When we multiply the given numbers, we find

$$(3-2i)(-2-3i) = -12-5i.$$

11.3 Problems (6 pt Problems)

1. Simplify $\sqrt{-20}$.
2. Divide $\dfrac{3-2i}{-2+3i}$.
3. Solve $2x(x+4) = -3$.

11.4 Exercises

1. Simplify $4\sqrt{-32}$.
2. Write in standard form: $-2+6i-(5+2i)$.
3. Write in standard form: $(-2+6i)(5+2i)$.
4. Write in standard form: $\dfrac{-2+6i}{-5-2i}$.
5. Solve $x^2-5x+20=0$.
6. Solve $3x(x+2) = 2x-5$.

12 Polynomial Equations

12.1 Discussion Problems

1. What does knowing that $(x - 2)$ is a factor of a polynomial tell us about the zeros of that polynomial? Give an example.
2. What does knowing that -1 is a zero of a polynomial tell us about its factors? Give an example.
3. If you know that $(3x - 2)$ is a factor of a polynomial, what does that tell you about the result of the division of that polynomial by $3x - 2$?

12.2 Example of the 6 point process

Find all solutions to $x^3 - 2x^2 + x + 4 = 0$.

- **Context**:

  Solving polynomial equation

  Solutions to equation

- **Observations**:

  The equation is cubic.

  There are three solutions (perhaps repeated).

  By trial and error, we see that -1 is a solution.

  There are 4 terms.

- **Questions**:

  What is a solution to an equation? What are we looking for?

  How do we solve a cubic equation?

  Are there similar problems? Solving a quadratic equation by factoring or the quadratic formula.

  What does knowing a solution tell us? How can we use that information to reduce the problem? We know that there is a factor $(x - (-1))$ or, $x + 1$.

  How do we find the other factor? Long division (just like with numbers)

  What form does the other factor take? It is quadratic.

  Can we use the fact that the other factor is quadratic to find more solutions? We can use the zero product property to find more solutions (either by factoring or using the quadratic formula).

  What form does our answer take? A list of numbers (at most 3).

- **Strategies**:

  We note that $x+1$ is a factor and perform long division on the polynomial on the left of the equal sign. The quotient is a quadratic. The zero product property tells us that zeros of this quadratic polynomial are solutions to the equation as well. We find the zeros by either factoring or using the quadratic formula.

- **Concepts**:

  Factor

  Zeros

  Solutions to equation

  Long division

  Quadratic equation

  Factoring

  Quadratic formula

  Possibly complex numbers

  Reducing fractions

  We see that -1 is a solution to the equation $x^3 - 2x^2 + x + 4 = 0$ so that $x+1$ is a factor of the polynomial on the right. Long division tells us that

  $$(x^3 - 2x^2 + x + 4)/(x+1) = (x^2 - 3x + 4)$$

  so that our equation is equivalent to

  $$(x+1)(x^2 - 3x + 4) = 0.$$

  The left side is a product that results in zero so one of the two factors must be zero. So either $x + 1 = 0$ or $x^2 - 3x + 4 = 0$. By using the quadratic formula, we find that if $x^2 - 3x + 4 = 0$ then

  $$\begin{aligned} x &= \frac{3 \pm \sqrt{9 - 4 \cdot 4}}{2} \\ &= \frac{3 \pm \sqrt{-7}}{2} \\ &= \frac{3 \pm i\sqrt{7}}{3} \\ &= 1 \pm \frac{\sqrt{7}}{3} i. \end{aligned}$$

  So, either $x = -1$ or $x = 1 \pm \frac{\sqrt{7}}{3} i$.

- **Conclusions**:

  So, we find that if $x^3 - 2x^2 + x + 4 = 0$ then $x = -1$, $x = 1 + \frac{\sqrt{7}}{3}i$, or $x = 1 - \frac{\sqrt{7}}{3}i$.

12.3 Problems (6 pt Problems)

1. Rewrite $x^3 - 3x - 2 = 0$ so that the polynomial on the left is in factored form. Use this form to solve the equation.
2. Solve $(x-2)(x^2 - 3x + 1) = 0$.
3. Solve $x(2x^2 + 5) = 5x^2 + 2$.

12.4 Exercises

1. Rewrite $x^3 + 5x^2 + 8x + 4 = 0$ so that the polynomial on the left is in factored form. Use this form to solve the equation.
2. Solve $(3x + 2)(2x^2 - x + 2) = 0$.
3. Solve $x^3 - 5x^2 + 8x - 4 = 0$ using the fact that 1 is a zero of $x^3 - 5x^2 + 8x - 4$.
4. Solve $x^3 - 2x^2 - 5x - 2 = 0$ (hint: -1 is one solution).

13 Rational Equations

13.1 Discussion Problems

1. What is a rational equation? Give an example that is not a polynomial equation.

2. What is a general strategy for solving rational equations?

3. What are 'extraneous solutions' and how do you recognize them?

4. Give one place that a rational equation might arise in an application.

13.2 Example of the 6 point process

Solve $\dfrac{3}{x-1} = \dfrac{x}{x+1}$.

- **Context**:

  Rational equation

  Solution to an equation

  Fractions

- **Observations**:

  The left and the right side are fractions with a variable in the denominator.

  There may be 'extraneous solutions'.

  There is one variable.

- **Questions**:

  What does it mean to solve an equation?

  What makes this different from a polynomial equation and is there a way to reduce this to a polynomial equation?

  What can we do with an equation to arrive at an almost equivalent equation?

  What are extraneous solutions?

  What is a general strategy for solving rational equations?

  What is the form of the solution?

- **Strategies**:

  Multiply both sides of the equation by the least common denominator which will give us a polynomial equation.

  Solve the polynomial equation and check to make sure solutions make sense in the original equation.

Alternatively, we could rewrite the equation so that there is a zero on one side and then simplify the fraction. Noting that a fraction is zero when the numerator is zero, we find the zeros of the numerator and check to see that they make sense in the original equation.

- **Concepts**:

  Fractions (arithmetic)

  Rational

  Equation

  Solution

  We begin our work by noting that the least common denominator of both sides of $\dfrac{3}{x-1} = \dfrac{x}{x+1}$ is $(x-1)(x+1)$. So multiplying both sides by $(x-1)(x+1)$ and reducing gives us

$$\frac{3(x+1)(x-1)}{x-1} = \frac{x(x+1)(x-1)}{x+1} \implies 3(x+1) = x(x-1)$$

  . We see that the result is a polynomial (quadratic) equation, so we arrange it in standard form so that we can use our techniques for solving quadratic equations.

$$\begin{aligned} 3(x+1) = x(x-1) &\implies 3x+3 = x^2 - x \\ &\implies x^2 - 4x - 3 = 0. \end{aligned}$$

  We fail at factoring, so we use the quadratic formula:

$$\begin{aligned} x &= \frac{4 \pm \sqrt{16 - 4(-3)}}{2} \\ &= \frac{4 \pm \sqrt{4 \cdot 7}}{2} \\ &= \frac{4 \pm 2\sqrt{7}}{2} \\ &= 2 \pm \sqrt{7}. \end{aligned}$$

  Both of these numbers make sense in the original equation and so these are both solutions.

- **Conclusions**:

  If $\dfrac{3}{x-1} = \dfrac{x}{x+1}$, then $x = 2 + \sqrt{7}$ or $x = 2 - \sqrt{7}$.

13.3 Problems (6 pt Problems)

1. Solve $\dfrac{x}{x+2} = \dfrac{x-1}{x+3}$.

2. Solve $\frac{1}{x} - \frac{2}{x-1} = \frac{x}{x^2 - x}$.

3. Suppose that a passenger train can travel 15 miles per hour faster than a certain freight train. If the passenger train can cover 400 miles in the same time that the freight train covers 250 miles, how fast is each train?

13.4 Exercises

1. Solve $\frac{x}{3} = \frac{x}{2} - 2$.

2. Solve $-\frac{2}{x+2} - 3 = \frac{x}{x+2}$.

3. Solve $\frac{x}{x^2 - 3x + 2} = \frac{2x}{x-2} + 1$.

4. Suppose it takes Ariane 8 hours to row back and forth to a bridge 6 miles away from her camp when the current is 1 mile an hour. How fast would she row in still water?

14 Radical Equations

14.1 Discussion Problems

1. How do you recognize and equation as a radical equation? Give an example.
2. What makes the radical equation different from any other equation you have seen so far in this course?
3. How can you solve $\sqrt{x} = 3$? How does this help you solve $\sqrt{x-2} = 3$?
4. How do your solutions to the above questions help you with a strategy to solve a radical equation? If not can you think of how to 'undo' a radical (recall the section on radicals and rational exponents).
5. Why doesn't the strategy work for solving $\sqrt{x} = -3$?
6. What are 'extraneous solutions'?
7. If two quantities are equal, are their squares also equal?

14.2 Example of the 6 point process

Solve $\sqrt{x-3} = x - 2$.

- **Context**:

  Equation

  Solve

  Radical

- **Observations**:

  The radical is alone on the left side of the equation

  There is one variable that appears twice.

  There may be extraneous solutions.

- **Questions**:

  What does it mean to be a solution? What are we looking for?

  Do we know how to solve any similar problems?

  Might there be extraneous solutions? How will we check?

  What is the order of operations tell us about evaluating each side of this equation?

  What form does our answer take?

- **Strategies**:

  Since the term with the radical is alone on the left side, we will square both sides of the equation. That will give us a polynomial equation (quadratic). We will then solve by putting it into standard form and using the quadratic formula or factoring. We will need to check the solutions.

- **Concepts**:

  Solution to an equation

  Square root

  Isolating the radical

  Squaring a radical

  Squaring both sides of an equation

  Distribution

  We will begin by squaring both sides of the equation:

  $$\sqrt{x-3} = x - 5 \implies x - 3 = (x-3)^2.$$

  Here we must distribute the right hand side to get

  $$x - 3 = (x-5)^2 \implies x - 3 = x^2 - 10x + 25.$$

  Since this is a quadratic equation, we will write it in standard form and try to factor or use the quadratic formula.

  $$x^2 - 11x + 28 = 0 \implies (x-7)(x-4) = 0$$

  So, $x = 7$ or $x = 4$. We see here that

  $$\sqrt{7-3} = 7 - 5$$

  but

  $$\sqrt{4-3} \neq 4 - 5.$$

- **Conclusions**:

  If $\sqrt{x-3} = x - 5$ then $x = 7$.

14.3 Problems (6 pt Problems)

1. Solve $\sqrt{x-2} = 7$.
2. Solve $\sqrt{x+1} - 5 = x - 4$.
3. Solve $2\sqrt{x+3} + x = 5$.

14.4 Exercises

1. Solve $\sqrt{w} = 8$.

2. Solve $\sqrt{w} = -8$.

3. Solve $\sqrt{w-2} = 8$.

4. Solve $2\sqrt{w-2} + w = 8$.

5. Solve $\sqrt{w-2} = 4 - \sqrt{w+2}$. (challenge)

6. If a boat is 22 meters long and has a displacement of 22 cubic meters then the largest that area the sails S can be to qualify for a race satisfies

$$\frac{22 + 1.25\sqrt{S} - 9.8\sqrt[3]{22}}{0.686} = 24.$$

   What is the largest that the area of the sails can be in cubic meters?

7. If a right triangle has hypotenuse 5 feet and the perimeter is 12 feet, what are the lengths of the legs of the triangle? Be sure to draw a picture, label it, and form an appropriate equation whose solution leads to an answer.

15 Equations with Two Variables and Lines

15.1 Discussion Problems

1. Give an example of an equation with two variables.
2. What does it mean to be a solution of an equation with two variables? Give an example of an equation, a solution and a 'non-solution'.
3. What form does a solution of an equation with two variables have? How can we represent a solution? Give an example.
4. How many solutions can an equation with two variables have?
5. Consider the equation $y + 2x = 6$. Is there a solution of the form $(2, *)$? What about $(-1, *)$? How many solutions are there? How can we represent all solutions?
6. Consider the equation $y + 2x = 6$. Is there a solution of the form $(*, 2)$? What about $(*, -1)$? How many solutions are there? How can we represent all solutions?
7. How many parameters does the equation $y = mx + b$ have? How many solutions do we need to determine the value of these parameters? Which particular solutions would be easiest to use to determine the two parameters?
8. How do we recognize an equation with two variables as one whose graph is a line?
9. If we know that $(2, 3)$ is a solution to a linear equation with slope 3, what is the y-coordinate of the solution of the form $(3, *)$?
10. When calculating the slope of a line, does it matter which two points you choose to use for your calculation? Explain and give an example.

15.2 Example of the 6 point process

Graph the equation $y - 2x = 4$.

- **Context**:

  Equation with two variables

  Solutions to an equation

  Graph

- **Observations**:

  This is a linear equation.

  This is in so-called standard form.

  It has two variables x and y.

- **Questions**:

  How can we graph a linear equation? There are different ways. We can find two solutions and plot these on a coordinate plane and draw the line passing through them. Or, we can rewrite this equation in standard form, identify the slope and y-intercept and use these to plot the graph.

  How do we find two solutions of a linear equation in standard form?

  Are there problems that are similar to this that we know how to solve?

  How do we represent the solutions on a coordinate plane?

  How do we plot a point on the coordinate plane?

  What is the form of our answer?

- **Strategies**:

  Since this is in standard form, we will seek two solutions to the equation. Since the coefficients on both sides of the equation are integers, it will be possible to find points that have integer coordinates and are easy to find on the coordinate plane.

- **Concepts**:

  Solution to an equation with two variables

  Coordinate plane

  Line

  Two points determine a line.

  Let's look for a solution of the form $(0, *)$ (the y-intercept):

  $$y - 2 \cdot 0 = 4 \implies y = 4,$$

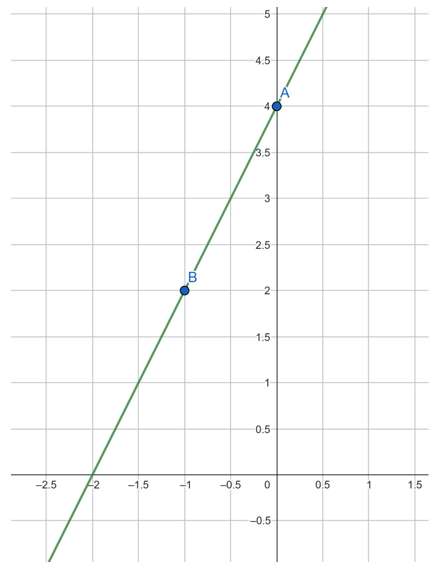
  so $(0, 4)$ is a solution.

  Now let's look for a solution of the form $(-1, *)$.

  $$y - 2(-1) = 4 \implies y + 2 = 4 \implies y + 2 - 2 = 4 - 2 \implies y = 2,$$

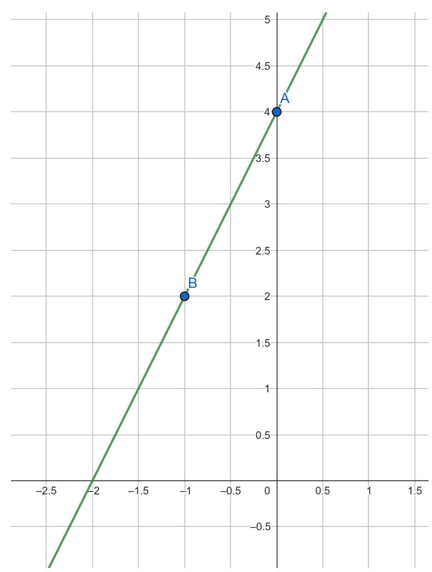
  so $(-1, 2)$ is a solution.

  Since the equation is linear, we can plot these points and then draw the line that passes through them to represent all of the solutions to this equation.



- **Conclusions**:

  We represent the solutions of the equation $y - 2x = 4$ by the graph



  where we understand the visible graph represents just some of the solutions. Other solutions can be imagined by extending the picture infinitely in space.

  We will check our graph by noting that it appears that $(-2, 0)$ is on our graph and should be a solution to the equation. And since $0 - 2(-2) = 4$

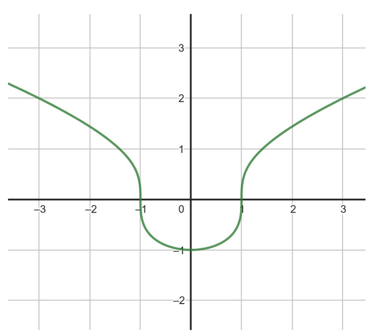
is true, it is a solution which supports the correctness of our graph. The point $(2,3)$ is not on our graph and so should not be a solution to our equation. Substituting into our equation gives $3-2(2)=4$, which is false, since the left hand side is -1, which is different from 4 so we confirm that $(2,3)$ is not a solution, which again supports the accuracy of the graph.

15.3 Problems (6 pt Problems)

1. Find a solution to the equation $3y-2x=-6$ with integer values.
2. Find an equation whose solution is represented by a line which passes through $(-2,1)$ and $(2,3)$.
3. Graph $y=-\frac{2}{3}x-2$.

15.4 Exercises

1. Is $(2,-1)$ a solution to the equation $x^3-y^3+y=3$? Is the point $(2,-1)$ on the graph of $x^3-y^3+y=3$?
2. Identify the slope and y-intercept of $y=-\frac{1}{2}x-1$, and graph the line.
3. Write the equation of the line passing through $(3,1)$ and $(15,-10)$.
4. Find two solution of the equation $y=2$ as an equation with two variables, and use them to represent all solutions on a coordinate plane.
5. Write an equation for a line perpendicular to $y=2x-1$ which passes through $(-2,1)$.
6. Are the following lines parallel: $2x-4y=7$ and $3x-5y=8$? Explain.
7. Is $(2,1)$ a solution to the equation whose graph is given below?



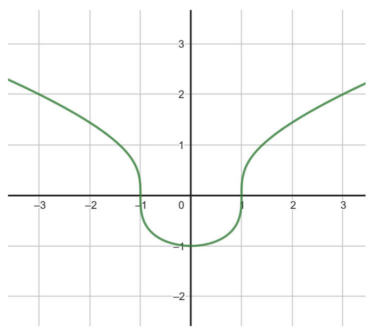
8. Find an equation representing the relationship between Celsius and Fahrenheit temperature scales noting the freezing point of water is 0°C and 32°F and boiling point of water is 100°C and 212°F. If it is 76°F outside, what is the temperature in Celsius (use your equation)?

9. A ladder is leaning against a wall so that it meets the wall 7 feet off the ground and the base of the ladder is 2 feet from the wall. If you orient yourself so that the slope of the ladder is positive, a safe slope is 3.87. Is your ladder safe to climb? Explain.

16 Quadratic Equations with Two Variables: Conics

16.1 Discussion Problems

1. How can you recognize a quadratic equation with two variables?
2. How can you recognize that an equation has a parabola as its graph?
3. How can you recognize that an equation has a circle as its graph?
4. Where is the vertex of the parabola whose equation is $y - 3 = 2(x + 1)^2$? Where is the center of the circle whose equation is $(x+1)^2 + (y-3)^2 = 4$? What is the similarity of these two questions?
5. How does the process of completing the square help to determine the location of your graph?
6. Suppose you have the graph



whose equation is $x^2 - y^3 = 1$. Graph the equation (without technology) $(x + 1)^2 - (y - 3)^3 = 1$.

7. Consider the equation $y = -x^2 + 7x - 1$. What does the sign of the coefficient of x^2 tell you about the graph?

16.2 Example of the 6 point process

Graph $x^2 + y^2 - 2x + 6y + 1 = 0$.

- **Context**:

  Equation with two variables

  Quadratic equation

  Solutions to equation

  Graph

- **Observations**:

  The coefficients of x^2 and y^2 are 1.

  The coefficients of x and y are not zero.

  There are five terms.

- **Questions**:

  What do solutions to quadratic equations look like?

  Conics: Circles, parabolas, hyperbolas, lines, hyperbolas

  What do the coefficients of the x^2 and y^2 terms tell us about the solutions? It helps us to know what to expect for a graph. In this case it is a circle.

  How many solutions are there? Infinitely many.

  Are there similar problems that we know how to solve? How is this one different? I know how to graph $x^2 + y^2 = 1$. Here there are no x or y terms.

  How will we transform our equation to a problem that we know something about? We will 'complete the square' twice.

  What is the form of our answer? A graph.

- **Strategies**:

  We will rewrite our equation so that the x's are together and the y's are together. Then we will complete the square and write it in 'standard form'. We will then be able to identify the center and radius of the circle. We will use this information to graph it.

- **Concepts**:

  Equation with two variables

  Quadratic equations

  Completing the square

  Translation of graph/equation

  Identifying parameters

  We first collect the x's and the y's:

  $$x^2 + y^2 - 2x + 6y + 1 = 0 \implies x^2 - 2x + y^2 + 6y + 1 = 0.$$

  We now seek to rewrite this so that the variable part looks like $(x-h)^2 + (y-k)^2$. But $(x-h)^2 = x^2 - 2hx + h^2$ so to make sure we have a $-2x$, we see $h = 1$. Similarly, and since $(y-k)^2 = y^2 - 2ky + k^2$ and we need to make sure we have a $6y$, we see that $k = -3$. So

  $$\begin{aligned} x^2 - 2x + y^2 + 6y + 1 = 0 &\implies (x^2 - 2x + 1) - 1 + (y^2 + 6y + 9) - 9 + 1 = 0 \\ &\implies (x-1)^2 + (y+3)^2 - 9 = 0 \\ &\implies (x-1)^2 + (y+3)^2 = 9. \end{aligned}$$

This looks like

$$x^2 + y^2 = 3^2,$$

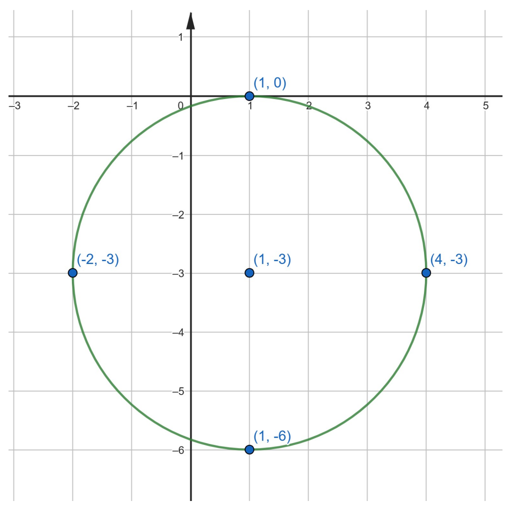
which is a circle of radius 3 centered at the origin.

The fact that all of the points on the circle are solutions to the original equation comes from the fact that the two equations are equivalent.

Our center is $(1, -3)$ since $(1 - 1) = 0$ and $(-3 + 3) = 0$ and the radius is 3. So we find the center and mark the four guiding points for our graph: 3 units up, down, left, and right of the circle (since these distances are easiest to see on the coordinate plane).

- **Conclusions**:

  Our circle has center $(1, -3)$ and radius 3. The graph (with four points labeled together with the center) follows.

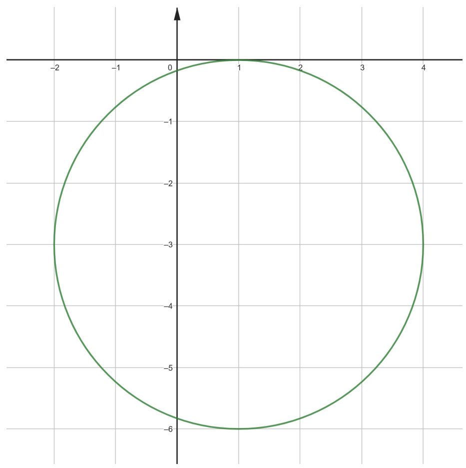


16.3 Problems (6 pt Problems)

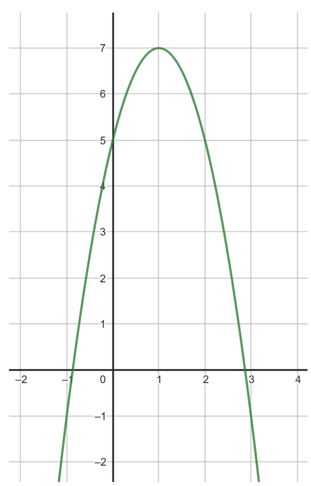
1. Graph $y + 2 = (x - 3)^2$.
2. Graph $y + x^2 - 4x + 3 = 0$.
3. Graph $x^2 + y^2 - 2x + 8y = 0$.
4. Suppose the height h in feet of an object at time t seconds is given by $h = -16t^2 + 32t + 2$. What is the maximum height of the object? Graph the equation and indicate the relationship between the graph and the solution to the problem.

16.4 Exercises

1. Graph $y - 1 = (x + 3)^2$ and $(x + 3)^2 + (y - 1)^2 = 16$.
2. Graph $x - 1 = (y^3)^2$.
3. Graph $y + x^2 + 6x - 1 = 0$.
4. Graph $x^2 + y^2 - 8x + 4 = 0$.
5. Find an equation whose graph is



6. Find an equation whose graph is



7. Suppose the height h in feet of an object at time t seconds is given by $h = -16t^2 + 16t$. How high is the object at 0 second? Use the equation to find the x-intercepts and sketch the graph. How high is the object after 1/4 seconds? At what other time is this the height of the object?

8. Do some online exploration and find at least one application of a parabola in real life different from falling objects.

17 Systems of Equations

17.1 Discussion Problems

1. What is a system of equations? Give an example with two variables.
2. What does it mean to be a solution to a system of equations with two variables? Give an example.
3. How do the graphs of the equations in a system of equations help us to estimate the solutions?
4. How many solutions could there be to a system of equations?
5. How do we check our solution?
6. What are the limitations and advantages to solving a system by graphing?
7. What are the limitations and advantages to solving a system by substitution?
8. What are the limitations and advantages to solving a system by elimination?
9. Give an example of when such system of equations might arise.

17.2 Example of the 6 point process

Solve the following system of equations:

$$\begin{cases} x - 2y &= -4 \\ 2x - 3y &= -5 \end{cases}$$

- **Context**:

  System of equations

  Solution

  Linear

- **Observations**:

  These are two linear equations with two variables.

  The x in the first equation has a coefficient of 1.

  There is more than one approach to solving this equation.

  These lines are not parallel.

- **Questions**:

  Is this like any other problems we have seen?

  Could this be reduced to a simpler, more familiar problem?

  How can we benefit from the x having leading coefficient 1?

  How will we keep track of our work?

  How many solutions are there? One.

  What is the form of our answer? It is a value for x and a value for y, or, an ordered pair of numbers.

- **Strategies**:

  We could graph these lines and determine the solution, but this may be misleading depending on the quality of our graph and the actual solution may involve fractions which would be difficult to determine from a graph.

  We could use substitution. Solve for x in the first equation and substitute the equivalent expression in for x in the second equation. This would leave us with a linear equation for y which we could solve. Once we know y we could substitute it in either equation to find x.

  We could use elimination. We would multiply the first equation by an appropriate number (easy to find since the coefficient of x is one) so that when we add the left sides of the equation and the right sides, the x variable cancels. We solve the resulting equation for y. Once we know y we could substitute it in either equation to find x.

- **Concepts**:

  System of equations

  Solution

  Solving linear equations

  We will try to avoid more challenging arithmetic by proceeding with the method of elimination. If we multiply both sides of the first equation by -2, then the coefficients of x in the two equations are opposites.

  $$\begin{cases} x - 2y &= -4 \\ 2x - 3y &= -5 \end{cases} \Longrightarrow \begin{cases} -2x + 4y &= 8 \\ 2x - 3y &= -5 \end{cases}$$

  Then adding the left sides and the right sides produces the equation (because we are adding equivalent quantities to both sides of the first equation):

  $$y = 3.$$

  Now that we know $y = 3$, we can use the fact that $x - 2y = -4$ to find $x - 2(3) = -4$ so that if we add 6 to both sides we have $x = 2$.

- **Conclusions**:

  If x and y satisfy

$$\begin{cases} x - 2y &= -4 \\ 2x - 3y &= -5 \end{cases},$$

  then $x = 2$ and $y = 3$.

  We can check our solution by substituting it in each of the two equations:

$$\begin{cases} 2 - 2 \cdot 3 &= -4 \\ 2 \cdot 2 - 3 \cdot 3 &= -5 \end{cases}$$

  both of which are true, so the solution is $(2, 3)$.

  Note: if the solutions are complicated, we should at least check on a graph if they are approximately correct.

17.3 Problems (6 pt Problems)

1. Solve

$$\begin{cases} 3x - y &= 1 \\ 2x - 3y &= -2 \end{cases}$$

2. Solve

$$\begin{cases} 3x - 5y &= 1 \\ 2x - 3y &= -5 \end{cases}$$

3. Suppose that two fireworks are to be launched three seconds apart. If the height in feet at t seconds of the first firework is given by $h = -16t^2 + 160t$ and the height in feet at t seconds of the second firework is $h = -16(t - 3)^2 + 160(t - 3)$, at what time will they be at the same height? What is their height at that time?

17.4 Exercises

1. Solve

$$\begin{cases} x &= 3y - 1 \\ 2x - 3y &= -2 \end{cases}$$

2. Solve

$$\begin{cases} 3x - 5y &= -1 \\ 2x - 3y &= -2 \end{cases}$$

3. Solve

$$\begin{cases} 3x - 6y &= 2 \\ 2x - 4y &= -3 \end{cases}$$

4. Solve

$$\begin{cases} 3x^2 - 2y &= 2 \\ 2x - 4y &= -3 \end{cases}$$

5. Solve

$$\begin{cases} x^2 + y^2 &= 2 \\ 2x^2 - 4y &= -3 \end{cases}$$

6. Solve

$$\begin{cases} x^2 + y^2 &= 2 \\ x^2 + y^2 &= 3 \end{cases}$$

7. If there is a solution which is 3 parts water to 1 part salt and another solution has 4 parts water to 1 part salt, how much of the first and second solutions should you combine to have 3 gallons of a 10 part water to a 3 part salt solution?

8. Suppose you and your friend go to a fruit stand which sells apples and bananas (by item and not weight). You buy 3 bananas and an apple and your friend buys two apples and 2 bananas. Your receipts show that you spent 2 dollars and your friend spent 3 dollars. How much money would I need to buy 2 bananas and 7 apples?

18 Basic Trigonometry

18.1 Discussion Problems

1. What are similar triangles? How can you tell when two triangles are similar? Give an example.
2. How are the sides of a right triangle related? Give an example.
3. What are two classes of special right triangles (you know the angles and lengths of sides)? Give an example of each.
4. Draw a right triangle and suppose that an acute angle of your triangle measures θ. Write down six ratios and name them (relative to the angle with measure θ).
5. What does it mean for two angles to be co-terminal? Give an example.
6. Consider the unit circle and a ray making an angle θ with the x-axis. What are the coordinates of the intersection of the unit circle and the ray called? What is the slope of the ray called?
7. How does radian measure relate to degree measure? Give an example.
8. Why is cos not a number but $\cos\left(\frac{\pi}{4}\right)$ is?
9. Why is $(\cos\theta)^2 + (\sin\theta)^2 = 1$ for all θ?

18.2 Example of the 6 point process

Find all solutions to $2\cos\theta = -1$ in $[0, 2\pi)$.

- **Context**:

  Equation with one variable

  Trigonometry

  Solutions in $[0, 2\pi)$

  Radians

- **Observations**:

  This equation involves $\cos\theta$.

  Solutions must be in $[0, 2\pi)$.

  There is a 2 multiplying $\cos\theta$.

- **Questions**:

  How is $\cos\theta$ defined? It is the x-coordinate of the intersection of the unit circle with the ray making an angle θ with the x-axis.

  Is $\cos\theta$ negative or positive?

Does this look similar to another equation we have seen? If so, how are they different? It looks like $2x = -1$ but there is a $\cos\theta$ instead of x.

How do we reduce this to a statement about $\cos\theta$? Divide both sides by 2.

How will we solve the resulting equation $\cos\theta = -\frac{1}{2}$?

What is the meaning of that interval?

What kind of picture might we be able to draw?

What is the form of our answer?

- **Strategies**:

  We solve for $\cos\theta$. Then we draw a picture of the unit circle and rays consistent with the equation. Hopefully we can use our knowledge of special triangles, or a calculator to gain information. We will be guided by our picture to find all solutions.

- **Concepts**:

  $\cos\theta$

  Linear equation

  Unit circle

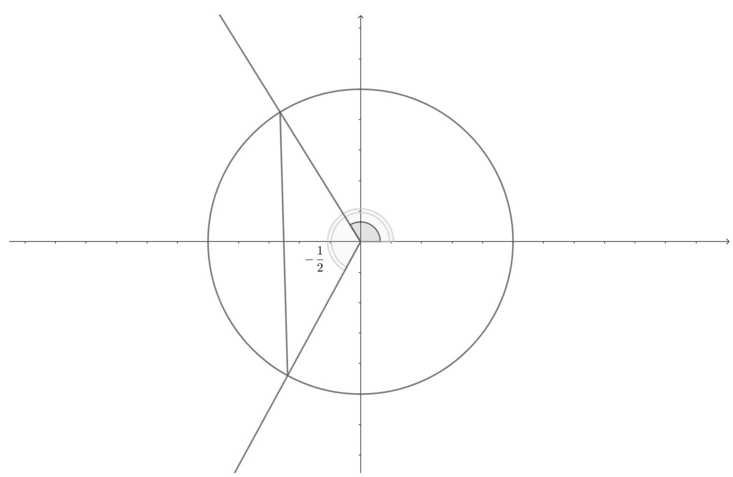
  Solving triangles/Special triangles

  Radians

  Fractions

Dividing both sides of the equation $2\cos\theta = -1$ by 2 gives $\cos\theta = -\frac{1}{2}$.

$\cos\theta$ is the x-coordinate of the intersection of the ray making an angle θ with the x-axis. We see that this is negative, so the x-coordinate of the intersection is negative. So the picture of the unit circle looks like:



where the two possible rays consistent with the equation and their corresponding angles are marked.

This forms two right triangles, and because we know the base of each triangle is $\frac{1}{2}$ (signed length is $\frac{-1}{2}$), we recognize this as a special triangle. The acute angle at the center of the circle of these triangles is $\frac{\pi}{3}$ or 60°.

It remains to determine the value of the two marked angles.

The smaller one is

$$\pi - \frac{\pi}{3} = \frac{2\pi}{3},$$

and the larger one is

$$\pi + \frac{\pi}{3} = \frac{4\pi}{3}.$$

- **Conclusions**:

  If $2\cos\theta = -1$ and θ is in $[0, 2\pi)$, then $\theta = \frac{2\pi}{3}$ or $\theta = \frac{4\pi}{3}$.

18.3 Problems (6 pt Problems)

1. Given the triangle $\triangle ABC$ with corresponding opposite sides a, b, and c, if B is a right angle and a has length 2 inches and c has length 3 inches, solve the triangle.

2. Evaluate $\tan\left(\frac{\pi}{6}\right)$.

3. Find all solutions to $\sin\theta = -\frac{1}{\sqrt{2}}$ which are in $[0, 2\pi)$.

18.4 Exercises

1. Given the triangle $\triangle ABC$ with corresponding opposite sides a, b, and c, if C is a right angle, a has length 2 inches and A has measure 27°, solve the triangle by estimating your answer using a special triangle and a calculator. All measurements to the nearest tenth.

2. The law of cosines states that for any triangle $\triangle ABC$ with corresponding opposite sides a, b, and c, you can choose an angle (we'll choose C) and the following is true about the measures (generalizing the case when C is a right angle):

   $$c^2 = a^2 + b^2 - 2ab\cos C.$$

   Use this to solve for c if $a = 3$, $b = 5$, and $\angle C = 42°$.

3. The law of sines states that for any triangle $\triangle ABC$ with corresponding opposite sides a, b, and c (associating the label with its measure):

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Use this to solve the triangle with $A = 92°$, $a = 20$ and $c = 15$.

4. Convert $80°$ to radians.

5. Convert $\frac{-3\pi}{10}$ radians to degrees.

6. Evaluate $\cos\left(\frac{7\pi}{6}\right)$.

7. Evaluate $\tan\left(\frac{8\pi}{3}\right)$.

8. Evaluate $\sin\left(\frac{-27\pi}{4}\right)$.

9. If $\sin\theta > 0$ and $\tan\theta = -2$, find all of the remaining trigonometric expressions relative to θ.

10. Find all solutions to $\tan\theta = -\frac{1}{\sqrt{2}}$ which are in $[0, 2\pi)$.

11. Suppose you have a 10 foot ladder. How far does the base need to be away from the wall so that the angle the ladder makes with the ground is $75.5°$?

12. If you are standing 100 feet away from a building and the angle of elevation is $40°$, how high is the building? (First estimate your answer using special triangles).

13. Use any resource to name three places where trigonometry shows up in applications.

19 Exponential and Logarithmic Expressions

19.1 Discussion Problems

1. What is the difference between x^2 and 2^x?
2. What is an exponential expression? Give an example.
3. How do you evaluate an exponential expression using a calculator? Give an example.
4. What is the meaning of $\log_2 8$?
5. Does $\log_5(-2)$ make sense? Explain.

19.2 Example of the 6 point process

Evaluate $\log_4 \left(\frac{1}{16}\right)$.

- **Context**:

  Logarithmic expression

  Evaluation

- **Observations**:

  The base of the logarithm is 4.

  There is a fraction.

- **Questions**:

  What does this logarithm mean? It is the exponent you need with base 4 to arrive at $\frac{1}{16}$.

  How do we arrive at fractions using exponents? We will need a negative exponent.

  What is the form of our answer? A number.

  How will we know if we are correct? We can check the evaluation of the related exponential expression.

- **Strategies**:

  We will write down a related exponential equation and solve by trial and error.

  Or we can rewrite $\frac{1}{16}$ as a power of 4 and use the definition of the logarithm.

- **Concepts**:
  Logarithm
  Exponential equation
  Base
  Exponent
  Negative exponent
  Fraction
  In evaluating $\log_4\left(\frac{1}{16}\right)$ we are looking for the exponent on 4 that will evaluate to $\frac{1}{16}$.
  So the value we are looking for is a solution to
  $$4^x = \frac{1}{16}.$$
  We might note that $4^2 = 16$ so that $4^{-2} = \frac{1}{16}$.
- **Conclusions**:
  We have found that $\log_4\left(\frac{1}{16}\right) = -2$.
  We can check our answer by evaluating 4^{-2}. Since this is $\frac{1}{16}$ our answer is correct.

19.3 Problems (6 pt Problems)

1. Evaluate 2^x at $0, 2, 4$ and 8. What do you notice?
2. Evaluate $\log_3 9$.
3. If we invest \$100 at an annual rate of 2% compounded quarterly, how much will we have after 5 years?

19.4 Exercises

1. Evaluate $3^x - 4^{2x}$ at $x = -1$ and at $x = 1$.
2. If we invest \$150 at an annual rate of 3% compounded quarterly, how much will we have after 10 years?
3. Using trial and error, how long would we have to invest \$150 at an annual rate of 3% compounded annually to have \$300? Can you write your answer as a logarithm?
4. Where else might you find applications of exponential expressions and logarithms. Give two examples.