Sinusoidal Alternating Waveforms

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Outlines

1. Intro. to Sinusoidal Alternating Waveforms
2. Frequency & Period
3. Phase Instantaneous
4. Peak & Peak-to-Peak
5. Average & Effective Values
6. AC Meters

Key Words: Sinusoidal Waveform, Frequency, Period, Phase, Peak, RMS, ac Meter

Sinusoidal Alternating Waveform is the time-varying voltage that is commercially available in large quantities and is commonly called the ac voltage. Each waveform in Fig. 13-1 is an alternating waveform available from commercial supplies. The term alternating indicates only that the waveform alternates between two prescribed levels in a set time sequence. To be absolutely correct, the term sinusoid, square-wave, or triangular must be applied.

Figure 13.1 Alternating waveforms.
Sinusoidal ac Voltage Generation

Sinusoidal ac voltages are available from a variety of sources. The most common source is the typical home outlet, which provides an ac voltage that originates at a power plant. In each case, an ac generator, as shown in Fig. 13-2(a), is primary component in the energy-conversion process. For isolated locations where power lines have not been installed, portable ac generators [Fig. 13-2(b)] are available that run on gasoline. The turning propellers of the wind-power station [Fig. 13-2(C)] are connected directly to the shaft of ac generator to provide the ac voltage as one of natural resources. Through light energy absorbed in the form of photons, solar cells [Fig. 13-2(d)] can generate dc voltage then can be converted to one of a sinusoidal nature through an inverter. Sinusoidal ac voltages with characteristics that can be controlled by the user are available from function generators, such as the one in Fig.13-2(e).

![Figure 13.2 Various sources of ac power; (a) generating plant; (b) portable ac generator; (c) wind-power station; (d) solar panel; (e) function generator.](image)

Sinusoidal ac Voltage Definitions

The sinusoidal waveform in Fig 13-3 with its additional notation will now be used as a model in defining a few basic terms. These terms, however, can be applied to any alternating waveform. It is important to remember, as you proceed through the various definitions, that the vertical scaling is in volts or amperes and the horizontal scaling is in units of time.

**Waveform:** The path traced by a quantity, such as the voltage in Fig. 13-3, plotted as a function of some variable such as time, position, degrees, radiations, temperature, and so on.

**Instantaneous value:** The magnitude of a waveform at any instant of time; denoted by lowercase letters ($e_1$, $e_2$ in Fig. 13-3)

**Peak amplitude:** The maximum value of a waveform as measured from its average, value, denoted by uppercase letters. For the waveform in Fig. 13-3, the average value is zero volts, and $E_m$ is defined by the figure.

**Peak-to-peak value:** Denoted by $E_{p-p}$ or $V_{p-p}$ (as shown in Fig. 13-3), the full voltage between positive and negative peaks of the waveform, that is, the sum of the magnitude of the positive and negative peaks.

**Periodic waveform:** A waveform that continually repeats itself after the same time interval. The Fig. 13-3 is a periodic waveform.

**Period (T):** The time of a periodic waveform.

**Cycle:** The portion of a waveform contained in one period of time. The cycles within $T_1$, $T_2$, and $T_3$ in Fig. 13-3 may appear different in Fig. 13-3, but they are all bounded by one period of time and therefore satisfy the definition of a cycle.

**Frequency (f):** The number of cycles that occur in 1 s. The frequency of the waveform in Fig. 13-5(a) is 1 cycle per second, and for Fig. 13-5(b), 2½ cycles per second. If a waveform of similar shape had a period of 0.5 s [Fig. 13-5 (c)], the frequency would be 2 cycles per second. 1 hertz (Hz) = 1 cycle per second (cps)

**Figure 13.3 Important parameters for a sinusoidal voltage.**

**Figure 13.4 Defining the cycle and period of a sinusoidal waveform.**

**Figure 13.5 Demonstration of the effect of a changing frequency on the period of a sinusoidal waveform.**
Ex. 13-1  For the sinusoidal waveform in Fig. 13-7.

a. What is the peak value?
b. What is the instantaneous value at 0.3 s and 0.6 s?
c. What is the peak-to-peak value of the waveform?
d. What is the period of the waveform?
e. How many cycles are shown?
f. What is the frequency of the waveform?

\[ V_{peak} = 8 \text{ V} \]
\[ v(0.3) = -8 \text{ V}, \quad v(0.6) = 0 \text{ V} \]
\[ V_{peak-to-peak} = 16 \text{ V} \]
\[ T = 0.4 \text{ s} \]
\[ n = 3.5 \text{ cycles} \]
\[ f = 2.5 \text{ cps, or } 2.5 \text{ Hz} \]

Ex. 13-2  Find the periodic waveform with a frequency of

a. 60 Hz  
\[ T = \frac{1}{f} = \frac{1}{60 \text{ Hz}} = 0.01667 \text{ s, or } 16.67 \text{ ms} \]

b. 1000 Hz  
\[ T = \frac{1}{f} = \frac{1}{1000 \text{ Hz}} = 0.001 \text{ s, or } 1 \text{ ms} \]

Ex. 13-3  Determine the frequency of the waveform in Fig. 13-9.

From the figure, \( T = (25 \text{ ms} – 5 \text{ ms}) \) or \( (35 \text{ ms} – 15 \text{ ms}) = 20 \text{ ms} \), and

\[ f = \frac{1}{T} = \frac{1}{20 \times 10^{-3} \text{ s}} = 50 \text{ Hz} \]
The unit of measurement for the horizontal axis can be **time**, **degree**, or **radians**. The term radian can be defined as follows: If we mark off a portion of the circumference of a circle by a length equal to the radius of the circle, as shown in Fig. 13-13, the angle resulting is called 1 radian. The result is

1 rad = 57.296° = 57.3°

where 57.3° is the usual approximation applied.

One full circle has 2π radians, as shown in Fig. 13-14. That is

2π rad = 360°

2π = 2(3.142) = 6.28

2π(57.3°) = 6.28(57.3°) = 359.84° = 360°

A number of electrical formulas contain a multiplier of π. For this reason, it is sometimes preferable to measure angles in radians rather than in degrees.

**The quantity is the ratio of the circumference of a circle to its diameter.**

\[
\text{Radians} = \left(\frac{\pi}{180}\right) \times \text{(degrees)}
\]

\[
\text{Degrees} = \left(\frac{180}{\pi}\right) \times \text{(radians)}
\]

For comparison purposes, two sinusoidal voltages are in Fig. 13-15 using degrees and radians as the units of measurement for the horizontal axis.

**Ex. 13-4** Determine the angular velocity of a sine wave having a frequency of 60 Hz.

\[
\omega = 2\pi f = (2\pi)(60 \text{ Hz}) \approx 377 \text{ rad/s}
\]

**Ex. 13-5** Determine the frequency and period of the sine wave in Fig. 13-17 (b).

\[
\omega = 2\pi / T,
\]

\[
T = \frac{2\pi}{\omega} = \frac{2\pi \text{ rad}}{500 \text{ rad} / \text{s}} = 12.57 \text{ ms}
\]

\[
\text{and } f = \frac{1}{T} = \frac{1}{12.57 \times 10^{-3} \text{s}} = 79.58 \text{ Hz}
\]
Ex. 13-6  Given \( \omega = 200 \text{ rad/s} \), determine how long it will take the sinusoidal waveform to pass through an angle of 90°.

\[ \alpha = \omega t, \text{ and } t = \frac{\alpha}{\omega} \]

However, \( \alpha \) must be substituted as \( \pi / 2 \) (= 90°)

\[ \sin \alpha = \sin \omega t \text{ is in radians per second} \]

\[ t = \frac{\alpha}{\omega} = \frac{\pi / 2 \text{ rad}}{200 \text{ rad/s}} = \frac{\pi}{400} \text{ s} = 7.85 \text{ ms} \]

Ex. 13-7  Find the angle through which a sinusoidal waveform of 60 Hz will pass in a period of 5 ms.

\[ \alpha = \omega t, \text{ or} \]

\[ \alpha = 2\pi f t = (2\pi)(60 \text{ Hz})(5 \times 10^{-3} \text{ s}) = 1.885 \text{ rad} \]

If not careful, you might be tempted to interpret the answer as 1.885°.

However, \( \alpha' = \frac{180\degree}{\pi \text{ rad}} (1.885 \text{ rad}) = 108\degree \)

Ex. 13-8  Given \( e = 5 \sin \alpha \), determine \( e \) at \( \alpha = 40\degree \) and \( \alpha = 0.8\pi \).

For \( \alpha = 40\degree \),

\[ e = 5 \sin 40\degree = 5(0.6428) = 3.21 V \]

For \( \alpha = 0.8\pi \),

\[ \alpha' = \frac{180\degree}{\pi \text{ rad}} (0.8\pi) = 144\degree \]

and \( e = 5 \sin 144\degree = 5(0.5878) = 2.94 V \)

Ex. 13-11  Given \( i = 6 \times 10^{-3} \sin 100t \), determine \( i \) at \( t = 2 \text{ ms} \).

\[ \alpha = \omega t = 1000t = (1000 \text{ rad/s})(2 \times 10^{-3} \text{ s}) = 2 \text{ rad} \]

\[ \alpha' = \frac{180\degree}{\pi \text{ rad}} (2 \text{ rad}) = 114.59\degree \]

\[ i = (6 \times 10^{-3}) (\sin 114.59\degree) = (6 \text{ mA})(0.9093) = 5.46 \text{ mA} \]

**General Format for the Sinusoidal Voltage or Current**

The basic mathematical format for the sinusoidal waveform is

\[ A_m \sin \alpha = A_m \sin \omega t \]

where \( A_m \) is the peak value of the waveform and \( \alpha \) is the unit of measure for the horizontal axis, as shown in Fig. 13-18.

For electrical quantities such as current and voltage, the general format is

\[ i = I_m \sin \omega t = I_m \sin \alpha \]

\[ e = E_m \sin \omega t = E_m \sin \alpha \]

where the capital letters with the subscript \( m \) represent the amplitude, and the lowercase letters \( I \) and \( e \) represent the instantaneous value of current and voltage at any time \( t \).

**Phase Relations**

If the waveform is shifted to the right or left of 0°, the expression becomes

\[ A_m \sin (\omega t \pm \theta) \]

where \( \theta \) is the angle in degrees or radians that the waveform has been shifted.

If the waveform passes through the horizontal axis with a positive going slope before 0°, as shown in Fig. 13-27, the expression is

\[ A_m \sin (\omega t + \theta) \]

If the waveform passes through the horizontal axis with a positive going slope after 0°, as shown in Fig. 13-28, the expression is

\[ A_m \sin (\omega t - \theta) \]

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**FIGURE 13.18** Basic sinusoidal function.

**FIGURE 13.27** Defining the phase shift for a sinusoidal function that crosses the horizontal axis with a positive going slope before 0°.

**FIGURE 13.28** Defining the phase shift for a sinusoidal function that crosses the horizontal axis with a positive going slope after 0°.
If the waveform crosses the horizontal axis with a positive-going slope 90° (π/2) sooner, as shown in Fig. 13-29, it is called a cosine wave; that is

$$\sin(\omega t + 90°) = \sin(\omega t + \pi/2) = \cos \omega t$$

or

$$\sin \omega t = \cos(\omega t - 90°) = \cos(\omega t - \pi/2)$$

$$\cos \alpha = \sin(\alpha + 90°)$$

$$\sin \alpha = \cos(\alpha - 90°)$$

$$-\sin \alpha = \sin(\alpha + 180°)$$

$$-\cos \alpha = \sin(\alpha + 270°) = \sin(\alpha - 90°)$$

$$\sin(-\alpha) = -\sin \alpha$$

$$\cos(-\alpha) = \cos \alpha$$

**FIGURE 13.29** Phase relationship between a sine wave and a cosine wave.

An oscilloscope can also be used to make phase measurements between two sinusoidal waveforms. Oscilloscopes have the dual-trace option, that is, the ability to show two waveforms at the same time. It is important that both waveforms must have the same frequency. The equation for the phase angle can be introduced using Fig. 13-37.

First, note that each sinusoidal waveform has the same frequency, permitting the use of either waveform to determine the period. For the waveform chosen in Fig. 13-37, the period encompasses 5 divisions at 0.2 ms/div. The phase shift between the waveforms is 2 divisions. Since the full period represents a cycle of 360°, the following division can be formed:

$$\frac{360°}{T(# of \ div.)} = \theta$$

$$\theta = \frac{\text{phase shift (# of \ div.)}}{T (# of \ div.)} \times 360°$$

and it leads i by 144°

**FIGURE 13.37** Finding the phase angle between waveforms using a dual-trace oscilloscope.

The oscilloscope is an instrument that will display the sinusoidal alternating waveform in a way that permit the reviewing of all of the waveform’s characteristics. The vertical scale is set to display voltage levels, whereas the horizontal scale is always in units of time.

**Ex. 13-13** Find the period, frequency, and peak value of the sinusoidal waveform appearing on the screen of the oscilloscope in Fig. 13-36. Note the sensitivities provided in the figure.

One cycle span 4 divisions. Therefore, the period is

$$T = 4 \text{ div} \times \frac{50 \mu\text{s}}{\text{div}} = 200 \mu\text{s}$$

and the frequency is

$$f = \frac{1}{T} = \frac{1}{200 \times 10^{-6} \text{s}} = 5 \text{kHz}$$

The vertical height above the horizontal axis encompasses 2 divisions. Therefore,

$$V_\alpha = 2 \text{ div} \times \frac{0.1 \text{ V}}{\text{div}} = 0.2 \text{ V}$$

**FIGURE 13.36** Average Value

The concept of the **average value** is an important one in most technical fields. In Fig. 13-38(a), the average height of the sand may be required to determine the volume of sand available. The average height of the sand is that height obtained if the distance from one end to the other is maintained while the sand is leveled off, as shown in Fig. 13-38(b). The area under the mound in Fig. 13-38(a) then equals the area under the rectangular shape in Fig. 13-38(b) as determined by $A = b \times h$. **FIGURE 13.38** Defining average value.  **FIGURE 13.39** Effect of distance (length) on average value. **FIGURE 13.40** Effect of depressions (negative excursions) on average value.
Ex. 13-14 Determine the average value of the waveforms in Fig. 13-42.

(a) Square wave

(b) Triangular wave

**FIGURE 13.42**

**a.** By inspection, the area above the axis equals the area below over one cycle, resulting in an average value of zero volts.

\[ G \text{ (average value)} = \frac{(10 \, V)(1 \, ms) + (-10 \, V)(1 \, ms)}{2 \, ms} = \frac{0}{2} = 0 \, V \]

**b.**

\[ G \text{ (average value)} = \frac{(14 \, V)(1 \, ms) + (-6 \, V)(1 \, ms)}{2 \, ms} = \frac{8 \, V}{2} = 4 \, V \]

Ex. 13-15 Determine the average value of the waveforms over one full cycle:

a. Fig. 13-44.

b. Fig. 13-45

**FIGURE 13.44**

**a.**

\[ G = \frac{(+3 \, V)(4 \, ms) + (-1 \, V)(4 \, ms)}{8 \, ms} = \frac{8 \, V}{8} = 1 \, V \]

**b.**

\[ G = \frac{(-10 \, V)(2 \, ms) + (4 \, V)(2 \, ms) + (-2 \, V)(2 \, ms)}{10 \, ms} = -16 \, V \text{ or } -1.6 \, V \]

Ex. 13-16 Determine the average value of the sinusoidal waveforms in Fig. 13-51.

**FIGURE 13.51**

The average value of a pure sinusoidal waveform over one full cycle is zero.

\[ G = \frac{(+A_0) + (-A_0)}{2 \pi} = 0 \, V \]

Ex. 13-17 Determine the average value of the waveforms in Fig. 13-52.

**FIGURE 13.52**

\[ G = \frac{(+2 \, mV) + (-16 \, mV)}{2} = -7 \, mV \]

Results in an average or dc level of \(-7 \, mV\), as noted by the dashed line in Fig. 13-52.

**Ex. 13-18 Effective (rms) Values**

This section begins to relate dc and ac quantities with respect to the power delivered to a load. The average power delivered by the ac source is just the first term, since the average value of a cosine wave is zero even though the wave may have twice the frequency of the original input current waveform. Equation the average power delivered by the ac generator to that delivered by the dc source.

\[ P_{dc} = P_{ac} \]

Which, in words, states that

**The equivalent dc value of a sinusoidal current or voltage is \(1 / \sqrt{2}\) or 0.707 of its peak value.**

The equivalent dc value is called the rms or effective value of the sinusoidal quantity.

\[ I_{rms} = \frac{1}{\sqrt{2}} I_m = 0.707 I_m \]

Similarly,

\[ E_{rms} = \frac{1}{\sqrt{2}} E_m = 0.707 E_m \]

\[ I_m = \sqrt{2} I_{rms} \]

\[ E_m = \sqrt{2} E_{rms} \]
Ex. 13-20 Find the rms values of the sinusoidal waveform in each part of Fig. 13-58.

(a) $I_{rms} = \frac{12 \times 10^{-3}}{\sqrt{2}} = 8.48\text{ mA}$

(b) $I_{rms} = 8.48\text{ mA}$

(c) $V_{rms} = \frac{169.73}{\sqrt{2}} = 120\text{ V}$

Note that frequency did not change the effective value in (b) compared to (a).

Ex. 13-21 The 120 V dc source in Fig. 13-59(a) delivers 3.6 W to the load. Determine the peak value of the applied voltage ($E_m$) and the current ($I_m$) if the ac source [Fig. 13-59(b)] is to deliver the same power to the load.

\[ P_{dc} = V_{dc} I_{dc} \quad \text{and} \quad I_{dc} = \frac{P_{dc}}{V_{dc}} = \frac{3.6\text{W}}{120\text{V}} = 30\text{ mA} \]

\[ I_m = \sqrt{2} I_{dc} = (1.414)(30\text{mA}) = 42.42\text{ mA} \]

\[ E_m = \sqrt{2} E_{dc} = (1.414)(120\text{V}) = 169.68\text{ mA} \]

Ac Meters and Instruments

It is important to note whether the DMM in use is a true rms meter or simply meter where the average value is calculated to indicate the rms level. A true rms meter reads the effective value of any waveform and is not limited to only sinusoidal waveforms.

Fundamentally, conduction is permitted through the diodes in such a manner as to convert the sinusoidal input of Fig. 13-68(a) to one having been effectively “flipped over” by the bridge configuration. The resulting waveform in Fig. 13-68(b) is called a full-wave rectified waveform.

Ex. 13-22 Find the rms value of the waveform in Fig. 13-60.

\[ V_{rms} = \sqrt{\frac{(3)^2(4)^2 + (-1)^2(4)^2}{8}} = \frac{40}{8} = 2.24\text{ V} \]

Ex. 13-24 Determine the average and rms values of the square wave in Fig. 13-64.

By inspection, the average value is zero.

\[ V_{rms} = \sqrt{\frac{(40)^2(10 \times 10^{-3})^2 + (40)^2(10 \times 10^{-3})^2}{20 \times 10^{-3}}} = \sqrt{\frac{32,000 \times 10^{-3}}{20 \times 10^{-3}}} = \sqrt{1600} = 40\text{ V} \]
Forming the ratio between the \textit{rms} and dc levels results in
\[
\frac{V_{\text{rms}}}{V_{dc}} = \frac{0.707V_m}{0.637V_m} \approx 1.11
\]

Meter indication = 1.11 (dc or average value) \textbf{Full-wave}

\textbf{Ex. 13-25} Determine the reading of each meter for each situation in Fig. 13-71(a) & (b).

For Fig. 13-71(a), situation (1):
Meter indication = 1.11(20V) = 22.2V

For Fig. 13-71(a), situation (2):
\[V_{\text{rms}} = 0.707V_m = 0.707(20V) = 14.14V\]

For Fig. 13-71(b), situation (1):
\[V_{\text{rms}} = V_a = 25V\]

For Fig. 13-71(b), situation (2):
\[V_{\text{rms}} = 0.707V_m = 0.707(15V) = 10.6V\]

\textbf{HW 13-37} Find the average value of the periodic waveform in Fig. 13.89.
\[G = \frac{(6V)(1s) + (3V)(1s) - (3V)(1s)}{3s}\]
\[= \frac{6V}{3} = 2V\]

\textbf{HW 13-42} Find the \textit{rms} value of the following sinusoidal waveforms:
\begin{enumerate}
  \item \(v = 140\sin(377t + 60^\circ)\)
  \item \(i = 6 \times 10^{-3} \sin(2\pi 1000t)\)
  \item \(v = 40 \times 10^{-6} \sin(2\pi 5000t + 30^\circ)\)
\end{enumerate}
\begin{enumerate}
  \item \(V_{\text{rms}} = 0.707(140V) = 98.99V\)
  \item \(I_{\text{rms}} = 0.707(6mA) = 4.24mA\)
  \item \(V_{\text{rms}} = 0.707(40\mu)V = 28.28\mu V\)
\end{enumerate}

\textbf{Homework 13:} 10-18, 30-32, 37, 42, 43
The response of the basic R, L, and C elements to a sinusoidal voltage and current are examined in this class, with special note of how frequency affects the “opposing” characteristic of each element. Phasor notation is then introduced to establish a method of analysis that permits a direct correspondence with a number of the methods, theorems, and concepts introduced in the dc chapter.

The derivative \( \frac{dx}{dt} \) is defined as the rate of change of \( x \) with respect to time. If \( x \) fails to change at a particular instant, \( dx = 0 \), and the derivative is zero. For the sinusoidal waveform, \( \frac{dx}{dt} \) is zero only at the positive and negative peaks (\( \omega t = \pi/2 \) and \( 3\pi/2 \) in Fig. 14-1), since \( x \) fails to change at these instants of time. The derivative \( dx/dt \) is actually the slope of the graph at any instant of time.

A close examination of the sinusoidal waveform will also indicate that the greatest change in \( x \) occurs at the instants \( \omega t = 0, \pi, \) and \( 2\pi \). The derivative is therefore a maximum at these points. At \( 0 \) and \( 2\pi \), \( x \) increases at its greatest rate, and the derivative is given positive sign since \( x \) increases with time. At \( \pi \), \( dx/dt \) decreases at the same rate as it increases at \( 0 \) and \( 2\pi \), but the derivative is given a negative sign since \( x \) decreases with time. For various values of \( \omega t \) between these maxima and minima, the derivative will exist and have values from the minimum to the maximum inclusive. A plot of the derivative in Fig. 14-2 shows that the derivative of a sine wave is a cosine wave.
The peak value of the cosine wave is directly related to the frequency of the original waveform. The higher the frequency, steeper the slope at the horizontal axis and the greater the value of $\frac{dv}{dt}$, as shown in Fig. 14-3 for two different frequencies. In addition, note that the derivative of a sine wave has the same period and frequency as the original sinusoidal waveform. For the sinusoidal voltage $e(t) = E_m \sin (\omega t \pm \theta)$, the derivative can be found directly by differentiation to produce the following:

$$\frac{d(e(t))}{dt} = \omega E_m \cos (\omega t \pm \theta)$$

For the sinusoidal voltage $e(t) = E_m \sin (\omega t)$, the derivative is $\frac{dv}{dt} = \omega E_m \cos (\omega t)$.

**FIGURE 14.3** Effect of frequency on the peak value of the derivative.

For the resistive element, the resistance $R$ in Fig. 14-4 can be treated as a constant, and Ohm’s law can be applied as follows. For $v = V_m \sin \omega t$,

$$\frac{v}{R} = \frac{V_m}{R} \sin \omega t$$

A plot of $v$ and $i$ in Fig. 14-5 reveals that for a purely resistive element, the voltage across and the current through the element are in phase, with their peak values related by Ohm’s law.

**FIGURE 14.5** Two voltage and current of a resistive element are in phase.

Response of Inductor to an ac Voltage or Current

For the series configuration in Fig. 14-6, the voltage $v_{element}$ of the boxed-in element opposes the source $e$ and thereby reduces the magnitude of the current $i$. The magnitude of the voltage across the element is determined by the opposition of the element to the flow of charge, or current $i$. For a resistive element, we have found that the opposition is its resistance and that $v_{element}$ and $i$ are determined by $v_{element} = iR$. The inductance voltage is directly related to the frequency and the inductance of the coil. For increasing values of $\omega$ and $L$ in Fig. 14-7, the magnitude of $v_L$ increases due the higher inductance and the greater the rate of change of the flux linkage. Using similarities between Figs. 14-6 and 14-7, we find that increasing levels of $v_L$ are directly related to increasing levels of opposition in Fig. 14-6. Since $v_L$ increases with both $\omega (= 2\pi f)$ and $L$, the opposition of an inductive element is as defined in Fig. 14-7.

**FIGURE 14.6** Defining the opposition of an element of the flow of charge through the element.

**FIGURE 14.7** Defining the parameters that determine the opposition of an inductive element to the flow of charge.
For the inductor in Fig. 14-8,

\[ v_L = L \frac{di_L}{dt} \]

and, applying differentiation,

\[ \frac{di_L}{dt} = L \omega \sin(\omega t) = \omega L \sin(\omega t) \]

Therefore,

\[ v_L = L \frac{di_L}{dt} = L(\omega L \cos(\omega t)) = \omega L \cos(\omega t) \]

or

\[ v_L = V_a \sin(\omega t + 90°) = V_a \cos(\omega t) \]

where

\[ V_a = \omega L I_a \]

Note that the peak value of \( v_L \) is directly related to \( \omega \) (= 2\pif) and \( L \) as predicted in the discussion previous slide. A plot of \( v_L \) and \( i_L \) in Fig. 14-9 reveals that for an inductor, \( v_L \) leads \( i_L \) by 90°.

If a phase angle is included in the sinusoidal expression for \( i_L \), such as

\[ i_L = I_a \sin(\omega t \pm \Theta) \]

then

\[ v_L = \omega L I_a \sin(\omega t \pm \Theta + 90°) \]

The opposition established by an inductor in a sinusoidal network is directly related to the product of the angular velocity and the inductance. The quantity \( \omega L \), called the reactance of an inductor, is symbolically represented by \( X_L \) and is measured in ohms.

That is,

\[ X_L = \omega L \quad \text{(ohms, } \Omega) \]

In an Ohm’s law format, its magnitude can be determined from

\[ X_L = \frac{V_a}{I_a} \quad \text{(ohms, } \Omega) \]

Inductive reactance is the opposition to the flow of current, which results in the continual interchange of energy between the source and the magnetic field of inductor. In other words, inductive reactance, unlike resistance, does not dissipate electrical energy.

Response of Capacitor to an ac Voltage or Current

For the capacitor, we will determine \( i \) for a particular voltage across the element. When this approach reaches its conclusion, we will know the relationship between the voltage and current and can determine the opposing voltage \( (v_{\text{element}}) \) for any sinusoidal current \( i \).

For capacitive networks, the voltage across the capacitor is limited by the rate at which charge can be deposited on, or released by, the plates of the capacitor during the charging and discharging phases, respectively. In other words, an instantaneous change in voltage across a capacitor is opposed by the fact that there is an element of time required to deposit charge on the plates of a capacitor, and \( V = Q/C \).

Since capacitance is a measure of the rate at which a capacitor will store charge on its plate,

for a particular change in voltage across the capacitor, the greater the value of capacitance, the greater the resulting capacitive current.

In addition, the fundamental equation relating the voltage across a capacitor to the current of a capacitor \( [i = C(dv/dt)] \) indicates that for particular capacitance, the greater the rate of change of voltage across the capacitor, the greater the capacitive current.

The current of a capacitor is therefore directly to the frequency and capacitance of the capacitor. An increase in either quantity results in an increase in the current of the capacitor. For the basic configuration in Fig. 14-10, we are interested in determining the opposition of the capacitor. Since an increase in current corresponds to a decrease in opposition, and \( i_c \) is proportional to \( \omega \) and \( C \), the opposition of a capacitor is inversely related to \( \omega \) and \( C \).

For the capacitor of Fig. 14-11,

\[ i_c = C \frac{dv_c}{dt} \]

and, applying differentiation,

\[ \frac{dv_c}{dt} = C \frac{d}{dt}(V_a \sin(\omega t)) = \omega C V_a \cos(\omega t) \]

Therefore,

\[ i_c = C \frac{dv_c}{dt} = C(\omega C V_a \cos(\omega t)) = \omega CV_a \cos(\omega t) \]

or

\[ i_c = I_a \sin(\omega t + 90°) \]

where

\[ I_a = \omega CV_a \]

Note that the peak value of \( v_c \) is directly related to \( C(= \text{farad}) \) and \( V_a \) as predicted in the discussion previous slide. A plot of \( v_c \) and \( i_c \) in Fig. 14-12 reveals that for a capacitor, \( v_c \) leads \( i_c \) by 90°.
A plot of $v_C$ and $i_C$ in Fig. 14-12 reveals that for a capacitor, $i_C$ leads $v_C$ by $90^\circ$.

If a phase angle is included in the sinusoidal expression for $v_C$, such as

$$v_C = V_m \sin(\omega t \pm \theta)$$

then

$$i_C = \omega CV_m \sin(\omega t \pm \theta + 90^\circ)$$

The quantity $1/\omega C$, called the reactance of a capacitor, is symbolically represented by $X_C$ and is measured in ohms; that is,

$$X_C = \frac{1}{\omega C} \quad \text{(ohms,} \ \Omega)$$

In an Ohm’s law format, its magnitude can be determined from

$$X_C = \frac{V_m}{I_m} \quad \text{(ohms,} \ \Omega)$$

**Ex. 14-1** The voltage across a resistor is indicated. Find the sinusoidal expression for the current if the resistor is 10 $\Omega$. Sketch the curves for $v$ and $i$.

a. $v = 100\sin(377t)$

b. $v = 25\sin(377t + 60^\circ)$

**Ex. 14-2** The current through a 5 $\Omega$ resistor is given. Find the sinusoidal expression for the voltage across the resistor for $i = 40\sin(377t + 30^\circ)$.

**Ex. 14-3** The current through a 0.1 H coil is provided. Find the sinusoidal expression for the voltage across the coil. Sketch the curves for $v$ and $i$ curves.

a. $i = 10\sin(377t)$

b. $i = 7\sin(377t - 70^\circ)$

**Ex. 14-4** The voltage across a 0.5 H coil is provided below. What is the sinusoidal expression for the current? Sketch the $v$ and $i$ curves.

$$v = 100\sin(20t)$$

**Ex. 14-5** The voltage across a 1 $\mu$F capacitor is provided below. What is the sinusoidal expression for the current? Sketch the $v$ and $i$ curves.

$$v = 30\sin(400t)$$

$X_L = \omega L = (377 \text{ rad/s})(0.1 \text{ H}) = 37.7 \Omega$

$V_m = I_mX_L = (10 \text{ A})(37.7 \Omega) = 377 \text{ V}$

and we know that for a coil $v$ leads $i$ by $90^\circ$.

Therefore,

$$v = 377 \sin(377t + 90^\circ)$$

$X_C = \omega C = (377 \text{ rad/s})(0.1 \text{ H}) = 37.7 \Omega$

$V_m = I_mX_C = (7 \text{ A})(37.7 \Omega) = 263.9 \text{ V}$

and we know that for a coil $v$ leads $i$ by $90^\circ$.

Therefore,

$$v = 263.9 \sin(377t - 70^\circ + 90^\circ)$$

and

$$v = 263.9 \sin(377t + 20^\circ)$$

$V_m = I_mR = (40 \text{ A})(5 \Omega) = 200 \text{ V}$

$X_L = \omega L = (377 \text{ rad/s})(0.1 \text{ H}) = 37.7 \Omega$

$V_m = I_mX_L = (12 \text{ A})(37.7 \Omega) = 2500 \text{ V}$

and we know that for a capacitor $i$ leads $v$ by $90^\circ$.

Therefore,

$$i = 12 \times 10^{-3} \sin(400t + 90^\circ)$$
Ex. 14-6 The current through a 100 μF capacitor is given. Find the sinusoidal expression for the voltage across the capacitor.

\[ i = 40 \sin(500t + 60°) \]

\[ X_c = \frac{1}{\omega C} = \frac{1}{(500 \text{ rad/s})(100 \times 10^{-6} \text{ F})} = \frac{10^9 \Omega}{5 	imes 10^7} = \frac{10^2 \Omega}{5} = 20 \Omega \]

\[ V_a = I_mC = (40 \text{ A})(20 \Omega) = 800 \text{ V} \]

and we know that for a capacitor \( V_a \) is 90° out of phase. Therefore,

\[ v = 800 \sin(500t + 60° - 90°) \]

\[ v = 800 \sin(500t - 30°) \]

Ex. 14-7 For the following pairs of voltage and currents, determine whether the element involved is a capacitor, an inductor, or a resistor. Determine the value of \( C \), \( L \), or \( R \) if sufficient data are provided (Fig. 14-18):.

a. \( V = 100 \sin(\omega t + 40°) \]
   \( i = 20 \sin(\omega t + 40°) \)

b. \( V = 1000 \sin(377t + 10°) \]
   \( i = 5 \sin(377t - 80°) \)

c. \( V = 500 \sin(157t + 30°) \]
   \( i = 1 \sin(157t + 120°) \)

d. \( v = 50 \cos(\omega t + 20°) \]
   \( i = 5 \sin(\omega t + 110°) \)

**Ideal Response**

**Resistor R**: For an ideal resistor, frequency will have absolutely no effect on the impedance level, as shown by the response in Fig. 14-19:

Note that 5 kHz or 20 kHz, the resistance of the resistor remain at 22 Ω; there is no change whatsoever. For the rest of the analyses in this text, the resistance level remains as the nameplate value; no matter frequency is applied.

**Inductor L**: For the ideal inductor, the equation for the reactance can be written as follows to isolate the frequency term in the equation. The result is a constant times the frequency variable that changes as we move down the horizontal axis of a plot:

\[ X_L = \frac{1}{\omega L} = \frac{1}{(500 \text{ Hz})(100 \text{ μF})} = \frac{1}{1 \text{ Ω}} = 22 \text{ Ω} \]

Thus far, each description has been for a set frequency, resulting in a fixed level of impedance for each of the basic elements. We must now investigate how a change in frequency affects the impedance level of the basic elements. It is an important consideration because most signals other than those provided by a power plant contain a variety of frequency levels.

**Frequency Response of the Basic Elements**

**Ideal Response**

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\[ X_L = \frac{1}{\omega L} = \frac{1}{(500 \text{ Hz})(100 \text{ μF})} = \frac{1}{1 \text{ Ω}} = 22 \text{ Ω} \]
Since a reactance of zero ohms corresponds with the characteristics of a short circuit, we can conclude that

at a frequency of 0 Hz, an inductor takes on the characteristics of a short circuit, as shown in Fig. 14-21.

![Figure 14.21](image1)

As shown in Fig. 14-21, as the frequency increases, the reactance increases, until it reaches an extremely high level at very high frequencies.

at very high frequencies, the characteristics of an inductor approach those of an open circuit, as shown in Fig. 14-21.

The inductor, therefore, is capable of handling impedance levels that cover the entire range, from ohms to infinite ohms, changing at a steady rate determined by the inductance level. The higher the inductance, the faster it approaches the open-circuit equivalent.

The result is that

at or near 0 Hz, the characteristics of a capacitor approach those of an open circuit, as shown in Fig. 14-23.

![Figure 14.23](image2)

As the frequency increases, the reactance approaches a value of zero ohms. The result is that

at very high frequencies, a capacitor takes on the characteristics of a short circuit, as shown in Fig. 14-23.

It is important to note in Fig. 14-22 that the reactance drops very rapidly as frequency increases. For capacitive elements, the change in reactance level can be dramatic with a relatively small change in frequency level. Finally, recognize the following:

As frequency increases, the reactance of an inductive element increases while that of a capacitor decreases, with one approaching an open-circuit equivalent as the other approaches a short-circuit equivalent.

The current through a 10 Ω capacitive reactance is given. Write the sinusoidal expression for the voltages. Sketch the v and i sinusoidal waveforms on the same set of axes.

a. \( i = 50 \times 10^{-3} \sin(\omega t) \)

b. \( i = 2 \times 10^{-6} \sin(\omega t + 60^\circ) \)

c. \( i = -6 \sin(\omega t - 30^\circ) \)

d. \( i = 3 \cos(\omega t + 10^\circ) \)

- \( v_a = I_a X_C = (50 \times 10^{-3}) \times (10 \Omega) = 0.5 \text{ V} \)
  \( v = 0.5 \sin(\omega t - 90^\circ) \)

- \( v_a = L_a X_C = (2 \times 10^{-6}) \times (10 \Omega) = 20 \mu\text{V} \)
  \( v = 20 \times 10^{-6} \sin(\omega t - 30^\circ) \)

- \( t = -6 \sin(\omega t - 30^\circ) = 6 \sin(\omega t + 150^\circ) \)
  \( V_a = L_a X_C = (6 \times 10^{-6}) \times (10 \Omega) = 60 \text{ V} \)
  \( v = 60 \sin(\omega t + 60^\circ) \)

- \( i = 3 \cos(\omega t + 100^\circ) \)
  \( V_a = L_a X_C = (3 \times 10^{-6}) \times (10 \Omega) = 30 \text{ V} \)
  \( v = 30 \sin(\omega t + 90^\circ) \)

**Capacitor C:** For the capacitor, the equation for the reactance

\[ X_C = \frac{1}{2\pi f C} \]

can be written as

\[ X_C f = \frac{1}{2\pi C} = k \quad \text{(a constant)} \]

which matches the basic format for a hyperbola

\[ xy = k \]

where \( X_C \) is the \( y \) variable, and \( k \) a constant equal to \( 1/(2\pi C) \)

![Figure 14.22](image3)

Hyperbolas have the shape appearing in Fig. 14-22 for two levels of capacitance. Note that the higher the capacitance, the closer the curve approaches the vertical and horizontal axes at low and high frequencies. At 0 Hz, the reactance of any capacitor is extremely high, as determined by the basic equation for capacitance:

\[ X_C = \frac{1}{2\pi f C} \approx \infty \Omega \]

**Homework 14.18:** The current through a 10 Ω capacitive reactance is given. Write the sinusoidal expression for the voltages. Sketch the v and i sinusoidal waveforms on the same set of axes.

- \( i = 50 \times 10^{-3} \sin(\omega t) \)
- \( i = 2 \times 10^{-6} \sin(\omega t + 60^\circ) \)
- \( i = -6 \sin(\omega t - 30^\circ) \)
- \( i = 3 \cos(\omega t + 10^\circ) \)
Average Power and Power Factor

A common question is, "How can a sinusoidal voltage or current deliver power to load if it seems to be delivering power during one part of its cycle and taking it back during the negative part of the sinusoidal cycle?" The equal oscillations above and below the axis seem to suggest that over one full cycle there is no net transfer of power or energy. However, there is a net transfer of power over one full cycle because power is delivered to the load at each instant of the applied voltage and current no matter what the direction is of the current or polarity of the voltage.

To demonstrate this, consider the relatively simple configuration in Fig. 14-29 where an 8 V peak sinusoidal voltage is applied across a 2 Ω resistor. When the voltage is at its positive peak, the power delivered at that instant is 32 W as shown in the figure. At the midpoint of 4 V, the instantaneous power delivered drops to 8 W; when the voltage crosses the axis, it drops to 0 W. Note that when the voltage crosses the its negative peak, 32 W is still being delivered to the resistor.

Figure 14.29 Demonstrating that power is delivered at every instant of a sinusoidal voltage waveform.
In total, therefore, 

**Even though the current through and the voltage across reverse direction and polarity, respectively, power is delivered to the resistive lead at each instant time.**

If we plot the power delivered over a full cycle, the curve in Fig. 14-30 results. Note that the applied voltage and resulting current are in phase and have twice the frequency of the power curve.

The fact that the power curve is always above the horizontal axis reveals that power is being delivered to the load at each instant of time of the applied sinusoidal voltage.

The average value of the second term is zero over one cycle, producing no net transfer of energy in any one direction. However, the first term in the preceding equation has a constant magnitude and therefore provides some net transfer of energy. This term is referred to as the average power or real power as introduced earlier. The angle \( \Theta_v - \Theta_i \) is the phase angle between \( v \) and \( i \). Since \( \cos(-\alpha) = \cos \alpha \), the magnitude of average power delivered is independent of whether \( v \) leads \( i \) or \( i \) leads \( v \).

**Resistor:** In a purely resistive circuit, since \( v \) and \( i \) are in phase, \( 1 \Theta_v - \Theta_i = \Theta = 0^\circ \), and \( \cos \Theta = \cos 0^\circ = 1 \), so that

\[
P = \frac{V I_m}{2} \cos \Theta (\text{watts}, W)
\]

where \( P \) is the average power in watts. This equation can also be written

\[
P = \left( \frac{V_m}{\sqrt{2}} \right) \left( \frac{I_m}{\sqrt{2}} \right) \cos \Theta
\]

or, since \( V_m = \frac{V}{\sqrt{2}} \) and \( I_m = \frac{I}{\sqrt{2}} \)

\[
P = \frac{V}{\sqrt{2}} \frac{I}{\sqrt{2}} \cos \Theta
\]

**Inductor:** In a purely inductive circuit, since \( v \) leads \( i \) by \( 90^\circ \), \( \Theta_v - \Theta_i = \Theta = 90^\circ \); therefore

\[
P = \frac{V I_m \cos 90^\circ}{2} = \frac{V I_m}{2} \cdot 0 = 0 \text{ W}
\]

The average power or power dissipated by the ideal inductor (no associate resistor) is zero watts.

**Capacitor:** In a purely capacitive circuit, since \( i \) leads \( v \) by \( 90^\circ \), \( \Theta_i - \Theta_v = \Theta = -90^\circ \); therefore

\[
P = \frac{V I_m \cos -90^\circ}{2} = \frac{V I_m}{2} \cdot 0 = 0 \text{ W}
\]

The average power or power dissipated by the ideal capacitor (no associate resistor) is zero watts.
Ex. 14-10 Find the average power dissipated in a network whose input current and voltage are the following:

\[ i = 5 \sin(\omega t + 40^\circ) \]
\[ v = 10 \sin(\omega t + 40^\circ) \]

Since \( v \) and \( i \) are in phase, the circuit appears to be purely resistive at the input terminals. Therefore,

\[ P = \frac{V_m I_m}{2} = \frac{(10V)(5A)}{2} = 25 W \]

or

\[ R = \frac{V_m}{I_m} = \frac{10V}{5A} = 2 \Omega \]

and

\[ P = \frac{V_m^2}{R} = \frac{(0.707)(10V)^2}{2} = 25 W \]

or

\[ P = \frac{I_m^2}{R} = \frac{(0.707)(5A)^2}{2} = 25 W \]

Ex. 14-11 Determine the average power delivered to networks having the following input voltage and current:

a. \( v = 100 \sin(\omega t + 40^\circ) \), \( i = 20 \sin(\omega t + 70^\circ) \)

\[ \theta = |\phi_i - \phi_v| = |40^\circ - 70^\circ| = 30^\circ \]

\[ P = \frac{V_m I_m \cos \theta}{2} = \frac{(100V)(20A) \cos(30^\circ)}{2} = \frac{(1000W)(0.866)}{2} = 866 W \]

b. \( v = 150 \sin(\omega t - 70^\circ) \), \( i = 3 \sin(\omega t - 50^\circ) \)

\[ \theta = |\phi_i - \phi_v| = |70^\circ - (-50^\circ)| = |20^\circ| = 20^\circ \]

\[ P = \frac{V_m I_m \cos \theta}{2} = \frac{(150V)(3A) \cos(20^\circ)}{2} = \frac{(450W)(0.9397)}{2} = 211.43 W \]
Ex. 14-12 Determine the power factors of the following loads, and indicate whether they are leading or lagging:

a. Fig. 14-35  
b. Fig. 14-36  
c. Fig. 14-37

Figure 14.35

Figure 14.36

Figure 14.37

The load is resistive, and the power factor is neither leading nor lagging.

Complex Numbers

In our analysis of dc network, we found it necessary to determine the algebraic sum of voltages and currents. Since the same will be also be true for ac networks, the question arises, How do we determine the algebraic sum of two or more voltages (or current) that are varying sinusoidally? Although one solution would be to find the algebraic sum on a point-to-point basis, this would be a long and tedious process in which accuracy would be directly related to the scale used.

It is purpose to introduce a system of complex numbers that, when related to the sinusoidal ac waveforms that is quick, direct, and accurate. The technique is extended to permit the analysis of sinusoidal ac networks in a manner very similar to that applied to dc networks.

A complex number represents a points in a two-dimensional plane located with reference to two distinct axes. This point can also determine a radius vector drawn from the original to the point. The horizontal axis called the real axis, while the vertical axis called the imaginary axis. Both are labeled in Fig. 14-38.

Two forms are used to represent a point in the plane or a radius vector drawn from the origin to that point.

Rectangular Form

The format for the rectangular form is

\[ C = X + jY \]

As shown in Fig. 14-39. The letter C was chosen from the word “complex.” The boldface notation is for any number with magnitude and direction. The italic is for magnitude only.

Ex. 14-13 Sketch the following complex numbers in the complex plane.

a. \( C = 3 + j4 \)  
b. \( C = 0 - j6 \)  
c. \( C = -10 - j20 \)
**Polar Form**

The format for the polar form is:

\[ C = Z \angle \theta \]

with the letter \( Z \) chosen from the sequence \( X, Y, Z \).

\( Z \) indicates magnitude only and \( \theta \) is always measured counterclockwise (CCW) from the positive real axis, as shown in Fig. 14-43. Angles measured in the clockwise direction from the positive real axis must have a negative sign associated with them. A negative sign in front of the polar form has the effect shown in Fig. 14-44. Note that it results in a complex number directly opposite the complex number with a positive sign.

**Conversion Between Forms**

The two forms are related by the following equations, as illustrated in Fig. 14-48.

- **Rectangular to Polar**
  
  \[ Z = \sqrt{X^2 + Y^2} \]
  
  \[ \theta = \tan^{-1} \frac{Y}{X} \]

- **Polar to Rectangular**
  
  \[ X = Z \cos \theta \]
  
  \[ Y = Z \sin \theta \]

---

**Ex. 14-14** Sketch the following complex numbers in the complex plane:

- \( C = 5 \angle 30^\circ \)
- \( C = 7 \angle -120^\circ \)
- \( C = -4.2 \angle 60^\circ \)

---

**Ex. 14-15** Convert the following from rectangular to polar form:

- \( C = 3 + j4 \) (Fig. 14-49)

  \[ Z = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5 \]
  
  \[ \theta = \tan^{-1} \left( \frac{4}{3} \right) = 53.13^\circ \]
  
  \[ C = 5 \angle 53.13^\circ \]

**Ex. 14-16** Convert the following from polar to rectangular form:

- \( C = 10 \angle 45^\circ \) (Fig. 14-50)

  \[ X = 10 \cos 45^\circ = (10)(0.707) = 7.07 \]
  
  \[ Y = 10 \sin 45^\circ = (10)(0.707) = 7.07 \]
  
  and \( C = 7.07 + j7.07 \)
Ex. 14-17 Convert the following from rectangular to polar form:
\[ C = -6 + j3 \]  
(Fig. 14-51)

\[ Z = \sqrt{(-6)^2 + (3)^2} = \sqrt{45} = 6.71 \]
\[ \beta = \tan \left( \frac{3}{-6} \right) = -26.57° \]
\[ \theta = 180° - 26.57° = 153.43° \]
\[ C = 5\angle153.43° \]

Ex. 14-18 Convert the following from polar to rectangular form:

\[ C = 10 \angle 230° \]  
(Fig. 14-52)

\[ X = 10 \cos 230° = -6.43 \]
\[ Y = 10 \sin 230° = -7.66 \]
and \[ C = -6.43 - j7.66 \]

HW 14-31 If the current through and voltage across an element are \[ i = 8 \sin(\omega t + 40°) \] and \[ v = 48 \sin(\omega t + 40°) \], respectively, compute the power by \[ P = I^2R \], \[ P = (V_m/I_m/2)\cos \theta \], and \[ P = VI\cos \theta \], and compare answers.

\[ R = \frac{V_m}{I_m} = \frac{48V}{8A} = 6 \Omega, \quad P = I^2R = \left( \frac{8A}{\sqrt{2}} \right)^2 6\Omega = 192 \text{ W} \]
\[ P = \frac{V_m I_m}{2} \cos \theta = \frac{(48V)(8A)}{\sqrt{2}} \cos 0° = 192 \text{ W} \]
\[ P = VI\cos \theta = \left( \frac{48V}{\sqrt{2}} \right) \left( \frac{8A}{\sqrt{2}} \right) \cos 0° = 192 \text{ W} \]
Acknowledgement

I want to express my gratitude to Prentice Hall giving me the permission to use instructor’s material for developing this module. I would like to thank the Department of Electrical and Telecommunications Engineering Technology of NYCity College of Technology (NYCCT) for giving me support to commence and complete this module. I hope this module is helpful to enhance our students’ academic performance.

Sunghoon Jang

Mathematical Operations with Complex Numbers

Complex numbers lend themselves readily to the basic mathematical operations of addition, subtraction, multiplication, and division. A few basic rules and definitions must be understood before considering these operations.

Let us first examine the symbol j associated with imaginary numbers. By definition,

\[ j = \sqrt{-1} \quad \text{Thus,} \quad j^2 = -1 \]

and \[ j^3 = j^2 \cdot j = -1 \cdot j = -j \]

with \[ j^4 = j^3 \cdot j = (-1)(-i) = +1 \]

\[ j^5 = j \]

and so on. Further,

\[ \frac{1}{j} = (1)(\frac{1}{j}) = \left( \frac{1}{j} \right) \left( \frac{j}{j^2} \right) = \left( \frac{j}{-1} \right) = -j \]
**Complex Conjugate:** The conjugate or complex conjugate of a complex number can be found by simply changing the sign of imaginary part in rectangular form or by using the negative of the angle of the polar form. For example, the conjugate of

\[ C = 2 + j3 \] is \[ 2 - j3 \]

as shown in Fig. 14 – 53. The conjugate of

\[ C = 2 \angle 30^\circ \] is \[ 2 \angle -30^\circ \]

as shown in Fig. 14 – 54.

**Reciprocal:** The reciprocal of a complex number is 1 divided by the complex number. For example, the reciprocal of

\[ C = X + jY \] is \[ \frac{1}{X + jY} \]

and of \( Z \angle \theta \),

\[ \frac{1}{Z \angle \theta} \]

We are now prepared to consider the four basic operations of addition, subtraction, multiplication, and division with complex numbers.

**Addition:** To add two or more complex numbers, add the real and imaginary parts separately. For example, if

\[ C_1 = X_1 + jY_1 \]
\[ C_2 = X_2 + jY_2 \]

then

\[ C_1 + C_2 = (X_1 + X_2) + j(Y_1 + Y_2) \]

There is really no need to memorize the equation if the alternative method of Example 14-20 is used.

**Subtraction:** In subtraction, the real and imaginary parts are again considered separately. For example, if

\[ C_1 = X_1 + jY_1 \]
\[ C_2 = X_2 + jY_2 \]

then

\[ C_1 - C_2 = (X_1 - X_2) + j(Y_1 - Y_2) \]

Again, there is really no need to memorize the equation if the alternative method of Example 14-20 is used.

**Example 14-19:**

a. Add \( C_1 = 2 + j4 \) and \( C_2 = 3 + j1 \)

b. Add \( C_1 = 3 + j6 \) and \( C_2 = -6 - j3 \)

\[ a. \ C_1 + C_2 = (2 + 3) + j(4 + 1) = 5 + j5 \]
\[ b. \ C_1 + C_2 = (3 - 6) + j(6 + 3) = -3 + j9 \]

**Example 14-20:**

a. Subtract \( C_2 = 1 + j4 \) from \( C_1 = 4 + j6 \)

b. Subtract \( C_2 = -2 + j5 \) from \( C_1 = 4 + j6 \)

\[ a. \ C_1 - C_2 = (4 - 1) + j(6 - 4) = 3 + j2 \]
\[ b. \ C_1 - C_2 = (3 - (-2)) + j(3 - 5) = 5 - j2 \]
Addition or subtraction cannot be performed in polar form unless the complex numbers have the same angle $\theta$ or unless they differ only by multiples of $180^\circ$.

**Multiplication:** To multiply two complex numbers in rectangular form, multiply the real and imaginary parts of one in turn by the real and imaginary parts of the other. For example, if

\[ C_1 = X_1 + jY_1 \quad \text{and} \quad C_2 = X_2 + jY_2 \]

then

\[ C_1 \cdot C_2 = (X_1 + jY_1)(X_2 + jY_2) \]

\[ = X_1X_2 + jY_1X_2 + jX_1Y_2 + j^2Y_1Y_2 \]

\[ = (X_1X_2 - Y_1Y_2) + j(Y_1X_2 + X_1Y_2) \]

and

\[ C_1 \cdot C_2 = (X_1X_2 - Y_1Y_2) + j(Y_1X_2 + X_1Y_2) \]

In Example 14-22(b), we obtain a solution without resorting to memorizing the equation above. Simply carry along the $j$ factor when multiplying each part of one vector with the real and imaginary parts of the other.

**Example 14-21**

- a. $2 \angle 45^\circ + 3 \angle 45^\circ = 5 \angle 45^\circ$. Note Fig. 14 – 59.
- b. $2 \angle 0^\circ - 4 \angle 180^\circ = 6 \angle 0^\circ$. Note Fig. 14 – 60.

**Example 14-22**

- a. Find $C_1 \cdot C_2$ if $C_1 = 2 + j3$ and $C_2 = 5 + j/10$
- b. Find $C_1 \cdot C_2$ if $C_1 = -2 - j3$ and $C_2 = +4 - j6$

- a. Using the format above, we have
  \[ C_1 \cdot C_2 = [(2)(3) - (3)(10)] + j[(3)(5) + (2)(10)] \]
  \[ = -20 + j35 \]
  Without using the format, we obtain
  \[ -2 - j3 \]
  \[ -4 - j6 \]
  \[ -8 - j12 \]
  \[ -8 + j(-12 + 12) = -8 \]
  and
  \[ C_1 \cdot C_2 = -26 \angle 180^\circ \]

**Ex. 14 – 23**

- a. Find $C_1 \cdot C_2$ if $C_1 = 5 \angle 20^\circ$ and $C_2 = 10 \angle 30^\circ$
- b. Find $C_1 \cdot C_2$ if $C_1 = 2 \angle 40^\circ$ and $C_2 = 7 \angle 120^\circ$

- a. $C_1 \cdot C_2 = (5 \angle 20^\circ)(10 \angle 30^\circ)$
  \[ = 5(10) \angle (20^\circ + 30^\circ) \]
  \[ = 50 \angle 50^\circ \]
  b. $C_1 \cdot C_2 = (2 \angle 40^\circ)(7 \angle 120^\circ)$
  \[ = (2)(7) \angle (-40^\circ + 120^\circ) \]
  \[ = 14 \angle 80^\circ \]

To multiply a complex number in rectangular form by a real number requires that both the real part and the imaginary part be multiplied by the real number. For example,

\[ (10)(2 + j3) = 20 + j30 \]

and

\[ 50 \angle 0^\circ(0 + j6) = j300 = 300 \angle 90^\circ \]
Division: To divide two complex numbers in rectangular form, multiply the numerator and denominator by the conjugate of the denominator and the resulting real and imaginary parts collected. That is, if

\[ \frac{C_1}{C_2} = \frac{X_1 + jY_1}{X_2 + jY_2} \]

then

\[ \frac{C_1}{C_2} = \frac{(X_1 + jY_1)(X_2 - jY_2)}{(X_2 + jY_2)(X_2 - jY_2)} \]

\[ \frac{C_1}{C_2} = \frac{(X_1X_2 + Y_1Y_2) + j(X_1Y_2 - X_2Y_1)}{X_2^2 + Y_2^2} \]

and

\[ \frac{C_1}{C_2} = \frac{X_1X_2 + Y_1Y_2 + j(X_1Y_2 - X_2Y_1)}{X_2^2 + Y_2^2} \]

The equation does not have to be memorized if the steps above used to obtain it are employed. That is, first multiply the numerator by the complex conjugate of the denominator and separate the real and imaginary terms. Then divide each term by the sum of each term of the denominator square.

Ex. 14-24

a. Find \( \frac{C_1}{C_2} \) if \( C_1 = 1 + j4 \) and \( C_2 = 4 + j5 \)

b. Find \( \frac{C_1}{C_2} \) if \( C_1 = -4 - j8 \) and \( C_2 = 6 - j1 \)

\[ \text{Ex. 14-25} \]

a. Find \( \frac{C_1}{C_2} \) if \( C_1 = 15 \angle 10^\circ \) and \( C_2 = 2 \angle 7^\circ \)

b. Find \( \frac{C_1}{C_2} \) if \( C_1 = 8 \angle 120^\circ \) and \( C_2 = 16 \angle -50^\circ \)

\[ \text{Phasors} \]

The addition of sinusoidal voltages and currents is frequently required in the analysis of ac circuits. One lengthy but valid method of performing this operation is to place both sinusoidal waveforms on the same set of axis and add algebraically the magnitudes of each at every point along the abscissa, as shown for \( c = a + b \) in Fig. 14-71. This, however, can be a long and tedious process with limited accuracy.
A shorter method uses the rotating radius vector. This radius vector, having a constant magnitude (length) with one end fixed at the origin, is called a phasor when applied to electric circuits. During its rotational development of the sine wave, the phasor will, at the instant \( t = 0 \), have the positions shown in Fig. 14-72(a) for each waveform in Fig. 14-72(b).

Note in Fig. 14-72(b) that \( v_2 \) passes through the horizontal axis at \( t = 0 \) s, requiring that the radius vector in Fig. 14-72(a) is equal to the peak value of the sinusoid as required by the radius vector. The other sinusoid has passed through 90° of its rotation by the time \( t = 0 \) s is reached and therefore has its maximum vertical projection as shown in Fig. 14-72(a). Since the vertical projection is a maximum, the peak value of the sinusoid that it generates is also attained at \( t = 0 \) s as shown in Fig. 14-72(b).

In general, for all of the analysis to follow, the phasor form of a sinusoidal voltage or current will be

\[
V = V\angle \theta \quad \text{and} \quad I = I\angle \theta
\]

where \( V \) and \( I \) are rms values and \( \theta \) is the phase angle. It should be pointed out that in phasor notation, the sine wave is always the reference, and the frequency is not represented.

Ex. 14-27 Convert the following from the time to the phasor domain:

<table>
<thead>
<tr>
<th>Time Domain</th>
<th>Phasor Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( \sqrt{2}(50)\sin(\omega t) )</td>
<td>( 50 \angle 0^\circ )</td>
</tr>
<tr>
<td>b. ( 69.9\sin((\omega t + 72^\circ)) )</td>
<td>( 50\angle 30^\circ )</td>
</tr>
<tr>
<td>c. ( (0.707)(45^\circ) )</td>
<td>( 31.82 \angle 90^\circ )</td>
</tr>
<tr>
<td>d. ( (0.707)(69.6) )</td>
<td>( 49.21 \angle 72^\circ )</td>
</tr>
</tbody>
</table>

Ex. 14-28 Write the sinusoidal expression for the following phasors if the frequency is 60 Hz:

<table>
<thead>
<tr>
<th>Time Domain</th>
<th>Phasor Domain</th>
</tr>
</thead>
</table>
| a. \( I = 10\angle 30^\circ \)  | \( i = \sqrt{2}(10)\sin(2\pi 60t + 30^\circ) \) 
| and \( i = 14.14\sin(377t + 30^\circ) \) |                                         |
| b. \( V = 115\angle 70^\circ \) | \( v = \sqrt{2}(115)\sin(377t - 30^\circ) \) 
| and \( v = 162.6\sin(377t - 30^\circ) \) |                                         |
Ex. 14-29  Find the input voltage of the circuit in Fig. 14-75 if

\[ v_a = 50 \sin(377t + 30°) \]
\[ v_b = 30 \sin(377t + 60°) \]

\[ f = 60 \text{ Hz} \]

Applying Kirchhoff’s voltage law, we have

\[ e_m = v_a + v_b \]

Converting from the time to the phasor domain yields

\[ v_a = 50 \sin(377t + 30°) \Rightarrow V_a = 35.35 V \angle 30° \]
\[ v_b = 50 \sin(377t + 60°) \Rightarrow V_b = 21.21 V \angle 60° \]

Converting from polar to rectangular form for addition yields

\[ V_a = 35.35 V \angle 30° = 30.61 V + j17.68 V \]
\[ V_b = 21.21 V \angle 60° = 10.61 V + j18.37 V \]

Ex. 14-30  Determine the current \( i_2 \) for the network in Fig. 14-77.

\[ i_2 = 120 \times 10^{-3} \sin(\omega t + 60°) \]

Applying Kirchhoff’s current law, we have

\[ i_2 = i_1 + i_3 \quad \text{or} \quad i_1 = i_2 - i_3 \]

Converting from the time to the phasor domain yields

\[ i_2 = 120 \times 10^{-3} \sin(\omega t + 60°) \Rightarrow 84.84 \text{ mA} \angle 60° \]
\[ i_3 = 80 \times 10^{-3} \sin \omega t \Rightarrow 56.56 \text{ mA} \angle 0° \]

Converting from polar to rectangular form for subtracting yields

\[ i_2 = 84.84 \text{ mA} \angle 60° = 42.42 \text{ mA} + j34.74 \text{ V} \]
\[ i_1 = 56.56 \text{ mA} \angle 0° = 56.56 \text{ mA} + j0 \]
HW 14-50 For the system in Fig. 14.87, find the sinusoidal expression for the unknown voltage $v_a$ if

\[ e_m = 60 \sin(377t + 20^\circ) \]

\[ v_b = 20 \sin(377t - 20^\circ) \]

(Using peak values)

\[ e_m = v_a + v_b \Rightarrow v_a = e_m - v_b = (60 \angle 20^\circ) - 20 \angle -20^\circ \]

\[ = 48.49 \angle 36.05^\circ \]

and \[ e_m = 46.49 \sin(377t + 36.05^\circ) \]

Figure 14.87 Problem 50.

HW 14-51 For the system in Fig. 14.88, find the sinusoidal expression for the unknown voltage $i_1$ if

\[ i_1 = 2 \times 10^{-6} \sin(\omega t + 60^\circ) \]

\[ i_2 = 6 \times 10^{-6} \sin(\omega t - 30^\circ) \]

\[ i_2 = i_1 + i_2 \Rightarrow i_1 = i_2 - i_2 \]

(Using peak values)

\[ (20 \times 10^{-6} A \angle 60^\circ) - (6 \times 10^{-6} A \angle -30^\circ) \]

\[ = 20.88 \times 10^{-6} A \angle 76.70^\circ \]

\[ i_1 = 20.88 \times 10^{-6} \sin(\omega t + 76.70^\circ) \]

Figure 14.88 Problem 51.

Homework 14: 39, 40, 43-45, 48, 50, 51
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Sunghoon Jang

OUTLINES

- Introduction to Series ac Circuits Analysis
- Impedance and Phase Diagram
- Series Configuration
- Voltage Divider Rule
- Frequency Response for Series ac Circuits

Key Words: Impedance, Phase, Series Configuration, Voltage Divider Rule

Series & Parallel ac Circuits

Phasor algebra is used to develop a quick, direct method for solving both series and parallel ac circuits. The close relationship that exists between this method for solving for unknown quantities and the approach used for dc circuits will become apparent after a few simple examples are considered. Once this association is established, many of the rules (current divider rule, voltage divider rule, and so on) for dc circuits can be applied to ac circuits.

Series ac Circuits

Impedance & the Phasor Diagram - Resistive Elements

From previous lesson we found, for the purely resistive circuit in Fig. 15-1, that \( V \) and \( I \) were in phase, and the magnitude

\[
I_m = \frac{V_m}{R} \quad \text{or} \quad V_m = I_m R
\]
In Phasor form, \( v = V_m \sin \omega t \Rightarrow V = V_{m} \angle 0^\circ \)

where \( V = 0.707 \times V_m \),

Applying Ohm's law and using phasor algebra, we have

\[
I = \frac{V}{R_{\angle 0^\circ}} = \frac{V_{m}}{R_{\angle 0^\circ}} \angle (0^\circ - \theta_{R})
\]

Since \( I \) and \( V \) are in phase, the angle associated with \( I \) also must be \( 0^\circ \).

To satisfy this condition, \( \theta_{R} \) must equal \( 0^\circ \). Substituting \( \theta_{R} = 0^\circ \), we found

\[
I = \frac{V_{m}}{R_{\angle 0^\circ}} \angle 0^\circ = \frac{V_{m}}{R} \angle (0^\circ - 0^\circ) = \frac{V_{m}}{R} \angle 0^\circ
\]

so that in the time domain,

\[
i = \sqrt{2} \left( \frac{V_{m}}{R} \right) \sin \omega t
\]

We use the fact that \( \theta_{R} = 0^\circ \) in the following polar format to ensure the proper phase relationship between the voltage and current of a resistor:

\[
Z_{R} = R_{\angle 0^\circ}
\]

**Ex. 15-2** Using complex algebra, find the voltage \( V \) for the circuit in Fig. 15-4.

Sketch the waveforms of \( V \) and \( i \).

Note Fig. 15-5:

\[
i = 4 \sin(\omega t + 30^\circ) \Rightarrow \text{phasor form } I = 2.828 A \angle 30^\circ
\]

\[
V = i Z_{R} = (1\angle 0^\circ)(2.828 A \angle 30^\circ)(2\angle 0^\circ) = 5.656 V \angle 30^\circ
\]

and

\[
v = \sqrt{2}(6.656)\sin(\omega t + 30^\circ) = 8.0\sin(\omega t + 30^\circ)
\]

**Ex. 15-1** Using complex algebra, find the current \( i \) for the circuit in Fig. 15-2.

Sketch the waveforms of \( V \) and \( i \).

Note Fig. 15-3:

\[
v = 100 \sin \omega t \Rightarrow \text{phasor form } V = 70.71 V \angle 0^\circ
\]

\[
I = \frac{V}{Z_{R}} = \frac{V_{\angle 0^\circ}}{R_{\angle 0^\circ}} = \frac{70.71 V \angle 0^\circ}{5 \Omega} = 14.14 A \angle 0^\circ
\]

and

\[
i = \sqrt{2}(14.14) \sin \omega t = 20 \sin \omega t
\]

Series ac Circuits

**Impedance & the Phasor Diagram – Inductive Elements**

From previous lesson we found that the purely inductive circuit in Fig. 15-7, voltage leads the current by \( 90^\circ \) and that the reactance of the coil \( X_L \) is determined by \( \omega L \).

\[
v = V_m \sin \omega t \Rightarrow \text{Phasor form } V = V_{m} \angle 0^\circ
\]

By ohm's law,

\[
I = \frac{V_{\angle 0^\circ}}{X_L \angle \theta_L} = \frac{V_{\angle 0^\circ}}{X_L \angle (0^\circ - \theta)}
\]

Since \( v \) leads \( i \) by \( 90^\circ \), \( i \) must have an angle of \( -90^\circ \) associated with it. To satisfy this condition, \( \theta_{L} \) must equal \( 90^\circ \). Substituting \( \theta_{L} = 90^\circ \), we obtain

\[
I = \frac{V_{\angle 0^\circ}}{X_L \angle 90^\circ} = \frac{V_{\angle 0^\circ}}{X_L \angle (0^\circ - 90^\circ)} = \frac{V_{\angle 0^\circ}}{X_L \angle -90^\circ}
\]

so that in the time domain,

\[
i = \sqrt{2} \left( \frac{V_{m}}{X_L} \right) \sin(\omega t - 90^\circ)
\]

We use the fact that \( \theta_{L} = 90^\circ \) in the following polar format for inductive reactance to ensure the proper phase relationship between the voltage and current of an inductor:

\[
Z_L = X_L \angle 90^\circ
\]
Ex. 15-3 Using complex algebra, find the current $i$ for the circuit in Fig. 15-8. Sketch the $v$ and $i$ curves.

Note: Fig. 15 - 9:

\[
v = 24\sin \omega t \Rightarrow \text{phasor form } V = 16.968 V \angle 0^\circ
\]

\[
I = \frac{V}{Z_L} = \frac{V \angle \theta}{X_L \angle 90^\circ} = \frac{16.968 V \angle 0^\circ}{3 \Omega \angle 90^\circ} = 5.656 A \angle -90^\circ
\]

and

\[
i = \sqrt{2}(5.656)\sin(\omega t - 90^\circ) = 8.0\sin(\omega t - 90^\circ)
\]

Figure 15.8 Example 15.3.

Figure 15.9 Waveform for Example 15.3.

Capacitive Resistance

For the pure capacitor in Fig. 15.13, the current leads the voltage by $90^\circ$ and that the reactance of the capacitor $X_C$ is determined by $1/\omega C$.

\[
X_C = \frac{1}{\omega C}
\]

Applying Ohm's law and using phasor algebra, we find

\[
I = \frac{V}{X_C \angle 0^\circ} = \frac{V}{X_C \angle (0^\circ - \theta_C)}
\]

Since $i$ leads $v$ by $90^\circ$, $i$ must have an angle of $+90^\circ$ associated with it. To satisfy this condition, $\theta_C$ must equal $-90^\circ$. Substituting $\theta_C = -90^\circ$ yields

\[
I = \frac{V}{X_C \angle -90^\circ} = \frac{V}{X_C \angle (-90^\circ)} = \frac{V}{X_C \angle 90^\circ}
\]

so, in the time domain,

\[
i = \sqrt{2}\left(\frac{V}{X_C}\right)\sin(\omega t + 90^\circ)
\]

Figure 15.13 Capacitive circuit.

We use the fact that $\theta_C = -90^\circ$ in the following polar format for capacitive reactance to ensure the proper phase relationship between the voltage and current of a capacitor:

\[
Z_C = X_C \angle -90^\circ
\]
Ex. 15-6 Using complex algebra, find the current \( v \) for the circuit in Fig. 15.16. Sketch the \( v \) and \( i \) curves.

\[ i = 6 \sin(\omega t - 60^\circ) \Rightarrow \text{phasor notation} \quad I = 4.242 \angle -60^\circ \]

\[ I = IZ_C = (I \angle \theta)(X_C \angle -90^\circ) = (4.242 A \angle -60^\circ) = (0.5\Omega \angle -90^\circ) = 2.121 \angle -150^\circ \]

\[ \text{and} \quad v = \sqrt{2}(2.121) \sin(\omega t - 150^\circ) = 3.0 \sin(\omega t - 150^\circ) \]

Ex. 15-7 Draw the impedance diagram for the circuit in Fig. 15.21, and find the total impedance.

The overall properties of series ac circuits (Fig. 15.20) are the same as those for dc circuits. For instance, the total impedance of a system is the sum of the individual impedances:

\[ Z_T = Z_1 + Z_2 + Z_3 + \cdots + Z_N \]

Ex. 15-8 Determine the input impedance to the series network in Fig. 15.23. Draw the impedance diagram.

\[ Z_T = Z_1 + Z_2 + Z_3 \]

\[ = R \angle 0^\circ + X_L \angle 90^\circ + X_C \angle -90^\circ \]

\[ = R + jX_L - jX_C \]

\[ = R + j(X_L - X_C) \]

\[ = 6\Omega + j(2\Omega - 12\Omega) \]

\[ = 6.32\Omega \angle -18.43^\circ \]

For the representative series ac configuration in Fig. 15.25 having two impedances, the currents is the same through each element (as it was for the series dc circuits) and is determined by Ohm’s law:

\[ I_T = Z_T \quad \text{and} \quad I = \frac{E}{Z_T} \]

The voltage across each element can be found by another application of Ohm’s law:

\[ V_1 = I_z z_1 \quad \text{and} \quad V_2 = I z_2 \]

KVL can then be applied in the same manner as it is employed for dc circuits. However, keep in mind that we are now dealing with the algebraic manipulation of quantities that have both magnitude and direction.

\[ -E + V_1 + V_2 = 0 \quad \text{or} \quad E = V_1 + V_2 \]

The power to the circuit can be determined by

\[ P = EI \cos \theta_T \]

where \( \theta_T \) is the phase angle between \( E \) and \( I \).
**Voltage Divider Rule**

The basic format for the **voltage divider rule** in ac circuits is exactly the same as that for dc circuits:

\[
V_x = \frac{Z_x E}{Z_T}
\]

where \(V_x\) is the voltage across one or more elements in a series that have total impedance \(Z_x\), \(E\) is the total voltage appearing across the series circuit, and \(Z_T\) is the total impedance of the series circuit.

**Ex. 15-9** Using the voltage divider rule, find the voltage across each element of the circuit in Fig. 15.40.

**Frequency Response for Series ac Circuits**

Thus far, the analysis has been for a fixed frequency, resulting in a fixed value for the reactance of an inductor or a capacitor. We now examine how the response of a series changes as the frequency changes. We assume ideal elements throughout the discussion so that the response of each element will be shown in Fig. 15.46.

When considering elements in series, remember that the total impedance is the sum of the individual elements and that the reactance of an inductor is in direct opposition to that capacitor. We now examine how the response of a series changes as the frequency changes. We assume ideal elements throughout the discussion so that the response of each element will be shown in Fig. 15.46.

In general, if we encounter a series R-L-C circuit at very low frequencies, we can assume that the capacitor, with its very large impedance, will be dominant factor. If the circuit is just an R-L series circuit, the impedance may be determined primarily by the resistive element since the reactance of the inductor is so small. As the frequency increases, the reactance of the coil increases to the point where it totally outshadows the impedance of the resistor. For an R-L-C combination, as the frequency increases, the reactance of the capacitor begins to approach a short-circuit equivalence, and total impedance will be determined primarily by the inductive element.

In total, therefore, when encountering a series circuit of any combination of elements, always use the idealized response of each element to establish some feeling for how the circuit will respond as the frequency changes.
As an example of establishing the frequency response of a circuit, consider the series R-C circuit in Fig. 15.47. As noted next to the source, the frequency range of interest is from 0 to 20 kHz.

The frequency at which the reactance of the capacitor drops to that of the resistor can be determined by setting the reactance of the capacitor equal to that of the resistor as follows:

\[ X_C = \frac{1}{2\pi fC} = R \]

Solving for the frequency yields

\[ f_1 = \frac{1}{2\pi RC} \]

Now for the details. The total impedance is determined by the following equation:

\[ Z_T = R - jX_C \]

and

\[ Z_T = Z_T \leq \theta_T = \sqrt{R^2 + X_C^2} - \tan^{-1}\frac{X_C}{R} \]  
(15.12)

The magnitude and angle of the total impedance can now be found at any frequency of interest by simply substituting into Eq. (15.12).

\[ f = 100 \text{ Hz} \]

\[ X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi (100 \text{ Hz})(0.01 \mu F)} = 159.16 \Omega \]

\[ X_T = \sqrt{R^2 + X_C^2} = \sqrt{(159.16\Omega)^2} = 159.24 \Omega \]

with \( \theta_T = -\tan^{-1}\frac{X_C}{R} = -\tan^{-1}\frac{88.2}{159.24} = -31.83^\circ = -88.2^\circ \)

and

\[ Z_T = 159.24\Omega \leq 88.2^\circ \]

\[ f = 1 kHz \]

\[ X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi (1kHz)(0.01 \mu F)} = 159.92 \Omega \]

\[ X_T = \sqrt{R^2 + X_C^2} = \sqrt{(159.92\Omega)^2} = 16.69 \Omega \]

with \( \theta_T = -\tan^{-1}\frac{X_C}{R} = -\tan^{-1}\frac{88.2}{159.92} = -31.83^\circ = -72.54^\circ \)

and

\[ Z_T = 16.69\Omega \leq 72.54^\circ \]

Ex. 15-12 For the series R-L circuit in Fig. 15.56:

a. Determine the frequency at which \( X_L = R \).

b. Develop a mental image of the change in total impedance with frequency without doing any calculations.

c. Find the total impedance at \( f = 100 \text{ Hz} \) and \( 40 \text{ kHz} \), and compare your answers with the assumptions of part (b)

d. Plot the curve of \( V_L \) versus frequency.

e. Find the phase angle of the total impedance at \( f = 40 \text{ kHz} \). Can the circuit be considered inductive at this frequency? Why?

a. \( X_L = 2\pi fL = R \) and

\[ f_1 = \frac{R}{2\pi L} = \frac{2k\Omega}{2\pi (40\text{mH})} = 7957.7 \text{ Hz} \]

6. At low frequencies, \( R > X_L \) and the impedance will be very close to that of the resistor, or \( 2k\Omega \). As the frequency increases, \( X_L \) increases to a point where it is the predominant factor. The result is that the curve starts almost horizontal at \( 2k\Omega \) and then increases linearly to very high levels.

We now that for frequencies greater than \( f_1, R > X_L \), and that for frequencies less than \( f_1, X_L > R \), as shown in Fig. 15.48.

Figure 15.47 Determining the frequency response of a series R-C circuit.

Figure 15.48 The frequency response for the individual elements of a series R-C circuit.

\[ V_L = \frac{Z_L}{Z_T} \]

From part (c), we know that at 100 Hz, \( Z_T = R \) so that \( V_L = V_C \) so that \( V_L = 20\text{V} \) and \( V_C = 0\text{V} \). The result is two plot points for the curve in Fig. 15.57.
At 1 kHz: \[ X_L = 2\pi fL \approx 0.25 \, \text{k}\Omega \]
and \[ V_L = \frac{(0.25 \, \text{k}\Omega \angle 90^\circ)(20V \angle 0^\circ)}{2\Omega + j0.25 \, \text{k}\Omega} \]
\[ = 2.48V \angle 82.87^\circ \]
At 5 kHz: \[ X_L = 2\pi fL \approx 1.26 \, \text{k}\Omega \]
and \[ V_L = \frac{(1.26 \, \text{k}\Omega \angle 90^\circ)(20V \angle 0^\circ)}{2\Omega + j1.26 \, \text{k}\Omega} \]
\[ = 10.68V \angle 57.79^\circ \]
At 10 kHz: \[ X_L = 2\pi fL \approx 2.5 \, \text{k}\Omega \]
and \[ V_L = \frac{(2.5 \, \text{k}\Omega \angle 90^\circ)(20V \angle 0^\circ)}{2\Omega + j2.5 \, \text{k}\Omega} \]
\[ = 15.63V \angle 38.66^\circ \]
The complete plot appears in Fig. 15.57.

The angle \( \theta_T \) is closing in on the 90° of a purely inductive network. Therefore, the network can be considered quite inductive at a frequency of 40 kHz.

HW 15-15 Calculate the voltage \( V_T \) and \( V_2 \) for the circuits in Fig. 15.134 in Phasor form using the voltage divider rule.

**Fig. 15.134 Problem 15:**

![Figure 15.134 Problem 15](image-url)

**Homework 15:**
2-7, 9-12, 15, 16
Parallel AC Circuits Analysis

Electrical and Telecommunications Engineering Technology Department

Professor Jang


Acknowledgement

I want to express my gratitude to Prentice Hall giving me the permission to use instructor’s material for developing this module. I would like to thank the Department of Electrical and Telecommunications Engineering Technology of NYCCT for giving me support to commence and complete this module. I hope this module is helpful to enhance our students’ academic performance.

Sunghoon Jang

OUTLINES

- Introduction to Parallel ac Circuits Analysis
- Impedance and Phase Diagram
- Parallel Configuration
- Current Divider Rule
- Frequency Response for Parallel ac Circuits
- Phase Measurements

Key Words: Parallel ac Circuit, Impedance, Phase, Frequency Response

Parallel ac Networks

For the representative parallel ac network in Fig. 15.67, the total impedance or admittance is determined as previously described, and the source current is determined by Ohm’s law as follows:

\[ I = \frac{E}{Z_T} = E Y_T \]

Figure 15.67 Parallel ac network.

Since the voltage is the same across parallel elements, the current through each branch can be found through another application of Ohm’s law:

\[ I_1 = \frac{E}{Z_1} = E Y_1 \quad \text{and} \quad I_2 = \frac{E}{Z_2} = E Y_2 \]

KCL can then be applied in the same manner as used for dc networks with consideration of the quantities that have both magnitude and direction.

\[ I - I_1 - I_2 = 0 \quad \text{or} \quad I = I_1 + I_2 \]

The power to the network can be determined by \( P = E I \cos \theta \)

where \( \theta \) is the phase angle between \( E \) and \( I \).
Parallel ac Networks: R-L

Phasor Notation:
As shown in Fig. 15.69.

\[ V_2 = Y_2 + Y_1 = G \angle 0^\circ + B_1 \angle 90^\circ \]
\[ = \frac{1}{1.67 \Omega} + \frac{1}{1.25 \Omega} \angle 90^\circ \]
\[ = 0.65 \angle 0^\circ + 0.8 \angle 90^\circ \]
\[ = 0.65 + j0.8 \angle 0^\circ \]
\[ = 1.05 \angle 53.13^\circ \]
\[ Z_1 = \frac{1}{Y_1} = \frac{1}{0.5 \angle 53.13^\circ} = 1 \Omega \angle -53.13^\circ \]

Admittance diagram: As shown in Fig. 15.70.

\[ I = \frac{E}{Z} = \frac{E_I}{(B_1 \angle 0^\circ)} \]
\[ = (20 \angle 0^\circ)(0.5 \angle 53.13^\circ) \]
\[ = 10 \angle 0^\circ \]
\[ = 10 \angle 0^\circ \]
\[ = 10 \angle 0^\circ \]
\[ = 10 \angle 0^\circ \]

Admittance diagram for the parallel R-L network in Fig. 15.68.

\[ P = \cos \theta_I \]
\[ = \cos 53.13^\circ \]
\[ = 0.6 \]
\[ = 60 W \]

KCL: At node a,
\[ I_1 - I_2 = 0 \text{ or } I_1 = I_2 \]
\[ 10A \angle 0^\circ = 6A \angle 53.13^\circ + 8A \angle -36.87^\circ \]
\[ 10A \angle 0^\circ = (6.60A + j4.80A) + (6.40A - j4.80A) \]
\[ = 10A + j0 \]
\[ \text{and} \]
\[ 10A \angle 0^\circ = 10A \angle 0^\circ \] (checks)

Power: The total power in watts delivered to the circuit is
\[ P = E \cos \theta_I \]
\[ = (20V)(10A) \cos 53.13^\circ = (200W)(0.6) \]
\[ = 120 W \]

Power factor: The power factor of the circuit is
\[ F_p = \cos \theta_I = \cos 53.13^\circ = 0.6 \text{ lagging} \]

Figure 15.68 Parallel R-L network.

Figure 15.69 Applying phasor notation to the network in Fig. 15.68.

Figure 15.70 Admittance diagram for the parallel R-L network in Fig. 15.68.

Parallel ac Networks: R-C

Phasor Notation:
As shown in Fig. 15.73.

\[ Y_2 = Y_2 + Y_1 = G \angle 0^\circ + B_1 \angle 90^\circ \]
\[ = \frac{1}{1.67 \Omega} + \frac{1}{1.25 \Omega} \angle 90^\circ \]
\[ = 0.65 \angle 0^\circ + 0.65 \angle 90^\circ \]
\[ = 0.65 + j0.65 \angle 0^\circ \]
\[ = 1.05 \angle 53.13^\circ \]
\[ Z_1 = \frac{1}{Y_1} = \frac{1}{0.5 \angle 53.13^\circ} = 1 \Omega \angle -53.13^\circ \]

Admittance diagram: As shown in Fig. 15.74.

\[ E = E_I \]
\[ = \frac{(10A \angle 0^\circ)(10V \angle 0^\circ)}{1} \]
\[ = 100 \angle 0^\circ \]
\[ = 100 \angle 0^\circ \]
\[ = 100 \angle 0^\circ \]
\[ = 100 \angle 0^\circ \]

Admittance diagram for the parallel R-C network in Fig. 15.72.

Figure 15.71 Phasor diagram for the parallel R-L network in Fig. 15.68.

Figure 15.72 Parallel R-C network.

Figure 15.73 Applying phasor notation to the network in Fig. 15.72.

Figure 15.74 Admittance diagram for the parallel R-C network in Fig. 15.72.

Parallel ac Networks: R-L-C

Phasor Notation:
As shown in Fig. 15.78.

\[ V_2 = V_2 + V_1 = G \angle 0^\circ + B_1 \angle 90^\circ \]
\[ = 1 \angle 0^\circ \]
\[ = 1 \angle 0^\circ \]
\[ = 1 \angle 0^\circ \]
\[ = 1 \angle 0^\circ \]

Figure 15.75 Parallel R-L-C network.

Figure 15.76 Applying phasor notation to the network in Fig. 15.75.

Figure 15.77 Parallel R-L-C network.

Figure 15.78 Applying phasor notation to the network in Fig. 15.77.
Admittance diagram: As shown in Fig. 15.79.
\[ I = \frac{E}{Z_t} = E Y_t = (100V \angle 53.13^\circ)(0.5S \angle -53.13^\circ) = 50A \angle 0^\circ \]
\[ I_1 = (E \angle 0^\circ)(G \angle 0^\circ) = (100V \angle 53.13^\circ)(0.3S \angle 0^\circ) = 30A \angle 53.13^\circ \]
\[ I_2 = (T \angle 0^\circ)(B \angle -90^\circ) = (100V \angle 53.13^\circ)(0.7S \angle -90^\circ) = 70A \angle -36.87^\circ \]
\[ I_C = (E \angle 0^\circ)(C \angle 0^\circ) = (100V \angle 53.13^\circ)(0.3S \angle +90^\circ) = 30A \angle 143.13^\circ \]

KCL: At node a,
\[ I_1 - I_2 - I_C = 0 \quad \text{or} \quad I_1 = I_2 + I_C \]

Power:
\[ P = E \cos \theta = (100V)(50A) \cos 53.13^\circ = (5000W)(0.6) = 3000 \, W \]

Power factor: The power factor of the circuit is
\[ \cos \theta = \cos 53.13^\circ = 0.6 \, \text{lagging} \]

Ex. 15-16 Using the current divider rule, find the current through each parallel branch in Fig. 15.83.
\[ I_1 = \frac{Z_1 I_1}{Z_1 + Z_2} \quad \text{or} \quad I_2 = \frac{Z_1 I_2}{Z_1 + Z_2} \]

In Fig. 15.85, the frequency response has been included for each element of a parallel R-L-C combination. At very low frequencies, the importance of the coil will be less than that of the resistor or capacitor, resulting in an inductive network in which the reactance of the inductor will have the most impact on the total impedance. As the frequency increases, the impedance of the inductor will increase while the impedance of the capacitor will decrease.
Let us now note the impact of frequency on the total impedance and inductive current for the parallel R-L network in Fig. 15.86 for a frequency range through 40 kHz.

In Fig. 15.87, \( X_L \) is very small at low frequencies compared to \( R \), establishing \( X_L \) as the predominant factor in this frequency range. As the frequency increases, \( X_L \) increases until it equals the impedance of resistor (220 \( \Omega \)). The frequency at which this situation occurs can be determined in the following manner:

\[
X_L = 2\pi f L = R \\
\text{and } f_2 = \frac{R}{2\pi L} = \frac{220\Omega}{2\pi(4 \times 10^{-3} H)} = 8.75 \text{ kHz}
\]

FIGURE 15.87 The frequency response of the individual elements of a parallel R-L network.

Another interesting development appears if the impedance of a parallel circuit, such as the one in Fig. 15.95(a), is found in rectangular form. In this case, the total impedance is equivalent to a capacitor with a reactance of 10 \( \Omega \), as shown in Fig. 15.94(b).

The total impedance at the frequency applied is equivalent to a capacitor with a reactance of 10 \( \Omega \), as shown in Fig. 15.94(b).

A general equation for the total impedance in vector form can be developed in the following manner:

\[
Z_T = \frac{RX_L}{\sqrt{R^2 + X_L^2}} \text{ and } \theta = 90^\circ - \tan^{-1} \frac{R}{X_L}
\]

FIGURE 15.94 Defining the equivalence between two networks at a specific frequency.

In a series circuit, the total impedance of two or more elements in series is often equivalent to an impedance that can be achieved with fewer elements of different values, the elements and their values being determined by frequency applied. This is also true for parallel circuits. For the circuit in Fig. 15.94 (a),

\[
Z_T = \frac{Z_L Z_C}{Z_L + Z_C} = \frac{(5\Omega - 90^\circ)(10\Omega - 90^\circ)}{5\Omega - 90^\circ + 10\Omega - 90^\circ} = \frac{50\Omega - 90^\circ}{5\Omega - 90^\circ} = 10\Omega - 90^\circ
\]

There is an alternative method to find same result by using formulas

\[
R_s = \frac{R X_L^2}{X_L^2 + R_s^2} = \frac{(3\Omega)^2(4\Omega)^2}{(4\Omega)^2 + (3\Omega)^2} = \frac{48\Omega}{25} = 1.92\Omega
\]

and

\[
X_s = \frac{X_L^2 X_C}{X_L^2 + X_C^2} = \frac{(3\Omega)^2(10\Omega)^2}{(4\Omega)^2 + (3\Omega)^2} = \frac{36\Omega}{25} = 1.44\Omega
\]

The magnitude of \( I_s \) is determined by \( I_s = \frac{R X_L}{\sqrt{R^2 + X_L^2}} \), and the phase angle \( \theta_s \), by which \( I_s \) leads \( I \), is given by

\[
\theta_s = -\tan^{-1} \frac{R}{X_L}
\]

FIGURE 15.99 Finding the series equivalent circuit for a parallel R-L network.

ET 242 Circuit Analysis II – Parallel ac circuits analysis

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Ex. 15-18 Determine the series equivalent circuit for the network in Fig. 15.97.

\[ R_p = 8k\Omega \]

\[ X_p(\text{resultant}) = |X'_p - X_c| = |9k\Omega - 4k\Omega| = 5k\Omega \]

\[ R_e = \frac{R_p X'_p}{X'_p + R_p} = \frac{(8k\Omega)(5k\Omega)^2}{(5k\Omega)^2 + (8k\Omega)^2} = \frac{200k\Omega}{89} = 2.25k\Omega \]

with \[ X'_p = \frac{R_p^2 X'_p}{X'_p + R_p} = \frac{(8k\Omega)^2(5k\Omega)^2}{(5k\Omega)^2 + (8k\Omega)^2} = \frac{320k\Omega}{89} = 3.6k\Omega \text{ (inductive)} \]

In Fig. 15.104, a resistor has been added to the configuration between the source and the network to permit measuring the current and finding the phase angle between the applied voltage and the source current. In Fig. 15.104, channel 1 is displaying the applied voltage, and channel 2 the voltage across the sensing resistor. Sensitivities for each channel are chosen to establishes the waveforms appearing on the screen in Fig. 15.105.

Using the sensitivities, the peak voltage across the sensing resistor is

\[ E_p = (4\text{div.})(2V/\text{div.}) = 8V \]

while the peak value of the voltage across the sensing resistor is

\[ V_{r,p(peak)} = (2\text{div.})(10mV/\text{div.}) = 20mV \]

For the chosen horizontal sensitivity, each waveform in Fig. 15.105 has a period \( T \) defined by ten horizontal divisions, and the phase angle between the two waveforms is \( 1.7 \) divisions. Using the fact that each period of a sinusoidal waveform encompasses \( 360^\circ \), the following ratios can be set up to determine the phase angle \( \theta \):

\[ \frac{\theta}{360^\circ} = \frac{1.7 \text{div.}}{10 \text{div.}} \] \[ \theta = \frac{1.7 \times 360^\circ}{10} = 61.2^\circ \]

In general,

\[ \theta = \left( \frac{\text{div. for } \theta}{\text{div. for } T} \right) 
\times 360^\circ \]

Therefore, the total impedance is

\[ Z_r = 4k\Omega \angle 61.2^\circ = 1.93k\Omega + j3.51k\Omega = R + jX_r \]

Phase Measurement

Measuring the phase angle between quantities is one of the most important functions that an oscilloscope can perform. Whenever you are using the dual-trace capability of an oscilloscope, the most important thing to remember is that both channels of a dual-trace oscilloscope must be connected to the same ground.

Measuring \( Z_r \) and \( \theta \)

For ac parallel networks, the total impedance can be found in the same manner as described for dc circuits: Simply remove the source and place an ohmmeter across the network terminals. However,

For parallel ac networks with reactive elements, the total impedance cannot be measured with an ohmmeter.

The phase angle between the applied voltage and the resulting source current is one of the most important because (a) it is also the phase angle associated with the total impedance; (b) it provides an instant indication of whether the network is resistive or reactive; (c) it reveals whether a network is inductive or capacitive; and (d) it can be used to find the power delivered to the network.
HW 15-25 Find the total admittance and impedance of the circuits in Fig. 15.142. Identify the values of conductance and susceptance, and draw the admittance diagram.

a. \( Z_T = 91 \angle 0^\circ = R \angle 0^\circ \), \( Y_T = 10.99 \text{ mS} \angle 0^\circ = G \angle 0^\circ \)

b. \( Z_T = 200 \angle 90^\circ = X_L \angle 90^\circ \), \( Y_T = 5 \text{ mS} \angle -90^\circ = B_L \angle -90^\circ \)

c. \( Z_T = 0.2 \text{ k}\Omega \angle -90^\circ = X_C \angle -90^\circ \), \( Y_T = 5.00 \text{ mS} \angle 90^\circ = B_C \angle 90^\circ \)

d. \( Z_T = \frac{(6 \Omega \angle 0^\circ)(60 \Omega \angle 90^\circ)}{10 \Omega + j60 \Omega} = 9.86 \Omega \angle 9.46^\circ = 9.73 \Omega + j1.62 \Omega = R + jX_C \)
\[ Y_T = 0.10 \text{ S} \angle -9.46^\circ = 0.1 \text{ S} - j0.02 \text{ S} = G - jB_L \]

e. \( Z_T = \frac{(2 \Omega \angle 0^\circ)(6 \Omega \angle -90^\circ)}{2 \Omega - j6 \Omega} = \frac{12 \Omega \angle -90^\circ}{6.32 \Omega \angle -71.57^\circ} = 1.90 \Omega \angle -18.43^\circ \)
\[ Y_T = 0.53 \text{ S} \angle 18.43^\circ = 0.5 \text{ S} - j0.17 \text{ S} = G + jB_C \]

f. \( Y_T = \frac{1}{3 \text{k}\Omega \angle 0^\circ} + \frac{1}{6 \text{k}\Omega \angle 90^\circ} + \frac{1}{9 \text{k}\Omega \angle -90^\circ} \)
\[ = 0.333 \times 10^{-3} \text{ S} \angle 0^\circ + 0.167 \times 10^{-3} \text{ S} \angle 90^\circ + 0.111 \times 10^{-3} \text{ S} \angle -90^\circ \]
\[ = 0.333 \times 10^{-3} \text{ S} - j0.056 \times 10^{-3} \text{ S} = 0.34 \text{ mS} \angle -9.55^\circ \]
\[ = G - jB_L \]
\[ Z_T = \frac{1}{Y_T} = 2.94 \text{k}\Omega \angle 9.55^\circ = 2.90 \text{k}\Omega + j0.49 \text{k}\Omega \]

Homework 15: 25, 27-32, 33, 39, 40, 47, 48
EET1222/ET242 Circuit Analysis II

Series & Parallel AC Circuits Analysis

Electrical and Telecommunication Engineering Technology

Professor Jang


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OUTLINES

- Introduction to Series - Parallel ac Circuits Analysis
- Reduction of series parallel Circuits to Series Circuits
- Analysis of Ladder Circuits

Key Words: ac Circuit Analysis, Series Parallel Circuit, Ladder Circuit

Series & Parallel ac Networks - Introduction

In general, when working with series-parallel ac networks, consider the following approach:

1. Redraw the network, using block impedances to combine obvious series and parallel elements, which will reduce the network to one that clearly reveals the fundamental structure of the system.

2. Study the problem and make a brief mental sketch of the overall approach you plan to use. In some cases, a lengthy, drawn-out analysis may not be necessary. A single application of a fundamental law of circuit analysis may result in the desired solution.

3. After the overall approach has been determined, it is usually best to consider each branch involved in your method independently before tying them together in series-parallel combinations. In most cases, work back from the obvious series and parallel combinations to the source to determine the total impedance of the network.

4. When you have arrived a solution, check to see that it is reasonable by considering the magnitudes of the energy source and the elements in the circuit.
For the network in Fig. 16.1:

a. Calculate $Z_T$.  
b. Determine $I_s$.  
c. Calculate $V_R$ and $V_C$.  
d. Find $I_C$.  
e. Compute the power delivered.  
f. Find $F_p$ of the network.

The total impedance is defined by:

$$Z_T = Z_1 + Z_2$$

with

$$Z_i = R + jX = \frac{1}{R} + j\frac{X}{R} = 80\angle 90\degree$$

Using the current divider rule yields

$$I = \frac{Z_1}{Z_1 + Z_2} = \frac{(80\angle 90\degree)(50\angle 30\degree)}{80\angle 90\degree + 50\angle 30\degree} = \frac{400\angle 60\degree}{30\angle 12\degree} = 13.33\angle 12\degree$$

b. $I_s = \frac{E}{Z_T} = \frac{120\angle 0\degree}{6.08\angle 80.54\degree} = 19.74\angle 80.54\degree$

c. Referring to Fig. 16.2, we find that $V_R$ and $V_C$ can be found by a direct application of Ohm’s law:

$$V_R = I_s Z_1 = (19.74\angle 80.54\degree)(1\Omega\angle 0\degree) = 19.74\angle 80.54\degree$$

$$V_C = I_s Z_2 = (19.74\angle 80.54\degree)(6\Omega\angle -90\degree) = 118.44\angle -9.46\degree$$

d. Now that $V_C$ is known, the current $I_C$ can be also found using Ohm’s law:

$$I_C = \frac{V_C}{Z_C} = \frac{118.44\angle -9.46\degree}{2\Omega\angle -90\degree} = 59.22\angle 80.54\degree$$

e. $P_{del} = I_s^2 R = (19.74\Omega^2)(1\Omega\angle 0\degree) = 389.67\ W$

f. $F_p = costh = cos 80.54\degree = 0.164$ leading

For the network in Fig. 16.3:

a. If $I$ is $50\ A\angle 30\degree$, calculate $I_1$ using the current divider rule.

b. Repeat (a) for $I_2$.

c. Verify Kirchhoff’s current law at one node.

Redrawing the circuit as in Fig. 16.4, we have

$$Z_1 = R + jX_1 = 3\Omega + j4\Omega = 5\Omega\angle 53.13\degree$$

$$Z_2 = 2\Omega + j\Omega = 8\Omega\angle -90\degree$$

Using the current divider rule yields

$$I_1 = \frac{Z_1}{Z_1 + Z_2} = \frac{(5\Omega\angle 53.13\degree)(50\angle 30\degree)}{5\Omega\angle 53.13\degree + 50\angle 30\degree} = \frac{250\angle 83.13\degree}{5\Omega\angle 53.13\degree + 50\angle 30\degree} = \frac{50\angle 136.26\degree}{5\Omega\angle 53.13\degree + 50\angle 30\degree}$$

$$I_2 = \frac{Z_2}{Z_1 + Z_2} = \frac{(2\Omega\angle -90\degree)(50\angle 30\degree)}{2\Omega\angle -90\degree + 50\angle 30\degree} = \frac{100\angle 60\degree}{2\Omega\angle -90\degree + 50\angle 30\degree} = \frac{5\angle 33.13\degree}{2\Omega\angle -90\degree + 50\angle 30\degree}$$

d. Now that $I_1$ is known, the current $I_2$ can be also found using Ohm’s law:

$$I_2 = \frac{V_1}{Z_2} = \frac{5\Omega\angle 0\degree}{2\Omega\angle -90\degree} = 2.54\angle -70\degree$$

$$I_2 = \frac{V_2}{Z_1} = \frac{5\angle 12\Omega\angle 0\degree}{5\Omega\angle 53.13\degree} = 1.54\angle 87.38\degree$$

and

$$I_2 = \frac{I_1 + I_2}{Z_1 + Z_2} = \frac{2.54\angle -70\degree + 1.54\angle 87.38\degree}{13\angle -62.38\degree} = \frac{4.08 \angle -72.5\degree + 0.07 \angle 11.34\degree}{13\angle -62.38\degree}$$

$$I_2 = 0.93 \angle -0.81 = 1.23\angle -41.05\degree$$
Ex. 16-4 For Fig. 16.7:

a. Calculate the current $I_c$.

b. Find the voltage $V_{ab}$.

Figure 16.9

Ex. 16-6 For the network in Fig. 16.12:

a. Determine the current $I$.

b. Find the voltage $V$.

c. The equivalent current source is their sum or difference (as phasors).

$I = 6mA \angle 20^\circ - 4mA \angle 0^\circ = 5.638mA + j2.052mA - 4mA$

$= 1.638mA + j0.052mA = 2.626mA \angle 51.40^\circ$

Figure 16.12

Ex. 16-7 For the network in Fig. 16.14:

a. Compute $I_c$.

b. Find $I_1$, $I_2$, and $I_3$.

c. Verify KCL by showing that $I = I_1 + I_2 + I_3$.

d. Find total impedance of the circuit.

Figure 16.15

Ex. 16-8 For the network in Fig. 16.16:

a. Calculate the total impedance $Z_T$.

b. Compute $I$.

c. Find the total power factor.

d. Calculate $I_1$ and $I_2$.

e. Find the average power delivered to the circuit.

Figure 16.17

Ex. 16-9 Compute $I_e$. Figure 16.18

$E = 200V \angle 0^\circ$  $I = 100V \angle 0^\circ$  $Z_e = 100^\circ$  $Z = 100^\circ$  $V_e = 100^\circ$

$I_e = \frac{E}{Z_e} = \frac{100V}{100^\circ} = 1A \angle 0^\circ$

$V = IZ = (1A \angle 0^\circ)(100^\circ) = 100V \angle 0^\circ$

Figure 16.19

ET 242 Circuit Analysis II – Series-Parallel Circuits Analysis

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Notice that all the desired quantities were conserved in the redrawn network. The total impedance:

\[ Z_T = Z_1 + Z_2 = 10 \Omega + 10 \Omega = 20 \Omega \]

Applying KCL yields

\[ I_T = I_1 - I_2 \]

\[ = (9.36 A - 1.5°) - (6.27 A - 36°) \]

\[ = (9.36 A - j0.25) - (5.07 - j3.69 A) \]

\[ = 4.29 A + j3.44 = 5.5 A \angle 38.72° \]

\[ \text{The total current:} \]

\[ I_T = 5.5 A \angle 38.72° \]

\[ \text{Voltage across the entire network:} \]

\[ V_T = 55 \text{ V} \angle 38.72° \]

\[ \text{Power delivered:} \]

\[ P_T = V_T I_T = (55 \text{ V} \angle 38.72°)(5.5 A \angle 38.72°) \]

\[ = (309.8 \text{ W} \angle 77.44°) \]

\[ \text{Therefore, the total power delivered is:} \]

\[ P_T = 309.8 \text{ W} \]

Ladder Networks

Ladder networks were discussed in some detail in Chapter 7. This section will simply apply the first method described in Section 7.6 to the general sinusoidal ac ladder network in Fig. 16.22. The current \( I_1 \) is desired.

Fig. 16.22 Ladder network.

Independence of \( Z_1 \), \( Z_3 \), and \( Z_5 \) and currents \( I_1 \) and \( I_3 \) are defined in Fig. 16.23.

\[ Z_T = Z_1 + Z_5 \]

\[ I_1 = \frac{E}{Z_T} \]

Fig. 16.23 Defining an approach to the analysis of ladder networks.

HW 16-13 Find the average power delivered to \( R_4 \) in Fig. 16.51.

\[ R_4 = 2.7 \Omega + 4.3 \Omega = 7 \Omega \]

\[ R' = 4 \Omega + 7 \Omega = 11 \Omega \]

\[ Z' = 2.1 \Omega - j10 \Omega \]

\[ (\text{CDR}) \quad I' = \frac{(49 \Omega \angle 0°)(20 mA \angle 0°)}{40 \Omega + 2.1 \Omega - j10 \Omega} = 19 mA \angle -0.014° \text{ as expected since } R_4 \gg Z' \]

\[ (\text{CDR}) \quad I_1 = \frac{(3 \Omega \angle 0°)(19 mA \angle 0.014°)}{3 \Omega + 7 \Omega} = 5.7 mA \angle 0.014° \]

\[ P = I_1^2 R = (5.7 mA)^2(4.3 \Omega) = 139.71 \text{ mW} \]

Homework 16: 1-8, 10, 12-14
Methods of Analysis and Selected Topics (AC)

For the networks with two or more sources that are not in series or parallel, the methods described previously cannot be applied. Rather, methods such as **mesh analysis** or **nodal analysis** to AC circuits must be used.

**Independent Versus Dependent (Controlled) Sources**

In the previous modules, each source appearing in the analysis of dc or ac networks was an **independent source**, such as $E$ and $I$ (or $V$ and $I$) in Fig. 17.1.

The term independent signifies that the magnitude of the source is independent of the network to which it is applied and that the source displays its terminal characteristics even if completely isolated.

A dependent or controlled source is one whose magnitude is determined (or controlled) by a current or voltage of the system in which it appears.

Currently two symbols are used for controlled sources. One simply uses the independent symbol with an indication of the controlling element, as shown in Fig. 17.2(a). In Fig. 17.2(a), the magnitude and phase of the voltage are controlled by a voltage $V$ elsewhere in the system, with the magnitude further controlled by the constant $k_1$. In Fig. 17.2(b), the magnitude and phase of the current source are controlled by a current $I$ elsewhere in the system, with the magnitude further controlled by $k_2$. To distinguish between the dependent and independent sources, the notation in Fig. 17.3 was introduced. Possible combinations for controlled sources are indicated in Fig. 17.4. Note that the magnitude of current sources or voltage sources can be controlled by voltage and a current.
### Source Conversions

When applying the methods to be discussed, it may be necessary to convert a current source to a voltage source, or a voltage source to a current source. This source conversion can be accomplished in much the same manner as for dc circuits, except dealing with phasors and impedances instead of just real numbers and resistors.

**Independent Sources**

In general, the format for converting one type of independent source to another is as shown in Fig. 17.5.

**Example 17.1**

Convert the voltage source in Fig. 17.6(a) to a current source.

\[
I = \frac{E}{Z} = \frac{100\angle 0^\circ}{5\Omega \angle 53.13^\circ} = 20\angle -53.13^\circ \quad [\text{Fig. 17.6(b)}]
\]

**Dependent Sources**

For the dependent sources, direct conversion in Fig. 17.5 can be applied if the controlling variable (V or I in Fig. 17.4) is not determined by a portion of the network to which the conversion is to be applied. For example, in Figs. 17.8 and 17.9, V and I, respectively, are controlled by an external portion of the network.

**Example 17.2**

Convert the current source in Fig. 17.9(a) to a voltage source.

\[
E = \frac{Z_L I}{Z_T + Z_L} = \frac{24\Omega \angle 0^\circ}{2\Omega + 24\Omega \angle 0^\circ} = 12\angle -90^\circ = 120\angle -30^\circ \quad [\text{Fig. 17.9(b)}]
\]

### Mesh Analysis

**Independent Voltage Sources**

The general approach to mesh analysis for independent sources includes the same sequence of steps appearing in previous module. In fact, throughout this section the only change from the dc coverage is to substitute impedance for resistance and admittance for conductance in the general procedure.

1. Assign a distinct current in the clockwise direction to each independent closed loop of the network.
2. Indicate the polarities within each loop for each impedance as determined by the assumed direction of loop current for that loop.
3. Apply KVL around each closed loop in the clockwise direction. Again, the clockwise direction was chosen to establish uniformity and to prepare us for the formed approach to follow.
   a. If an impedance has two or more assumed currents through it, the total current through the impedance is the assumed current of the loop in which KVL law is being applied.
   b. The polarity of a voltage source is unaffected by the direction of the assigned loop currents.
4. Solve the resulting simultaneous linear equations for the assumed loop currents.

The technique is applied as above for all networks with independent sources or for networks with dependent sources where the controlling variable is not a part of the network under investigation.
Ex. 17-5 Using the general approach to mesh analysis, find the current $I_1$ in Fig. 17.10.

The network is redrawn in Fig. 17.11 with subscripted impedances:

$Z_1 = +jX_1 = +j2\Omega$ \hspace{1em} $E_1 = 2V, \angle 0^\circ$

$Z_2 = R = 4\Omega$ \hspace{1em} $E_2 = 6V, \angle 0^\circ$

$Z_3 = -jX_3 = -j1\Omega$

Steps 1 and 2 are as indicated in Fig. 17.11.

Step 3:

$+ E_2 - I_1Z_1 - I_2Z_2 = 0$

$-Z_3(I_1 - I_2) - I_2Z_3 - E_3 = 0$

or

$E_1 - I_1Z_1 - I_2Z_2 + I_3Z_3 = 0$

$- I_1Z_1 + I_2Z_2 - I_3Z_3 - E_3 = 0$

so that

$I_1(Z_1 + Z_2) - I_2Z_2 = E_1$

$I_3(Z_1 + Z_3) - I_2 Z_3 = -E_2$

which are rewritten as

$I_1(Z_1 + Z_3) - I_2Z_2 = E_1$

$I_2Z_2 + I_3(Z_3 + Z_1) = -E_2$

Boylestad

Ex. 17-6 Write the mesh currents for the network in Fig. 17.12 having a dependent voltage source.

Ex. 17-7 Write the mesh currents for the network in Fig. 17.13 having an independent current source.

Dependent Voltage Sources

For dependent voltage sources, the procedure is modified as follows:

1. Step 1 and 2 are the same as those applied for independent sources.
2. Step 3 is modified as follows: Treat each dependent source like an independent when KVL is applied to each independent loop. However, once the equation is written, substitute the equation for the controlling quantity to ensure that the unknowns are limited solely to the chosen mesh currents.
3. Step 4 is as before.

Independent Current Sources

For independent current sources, the procedure is modified as follow:

1. Step 1 and 2 are the same as those applied for independent sources.
2. Step 3 is modified as follows: Treat each current source as an open circuit and write the mesh equations for each remaining independent path. Then relate the chosen mesh currents to the dependent sources to ensure that the unknowns for each equation are limited to the mesh currents.
3. Step 4 is as before.

Dependent Current Sources

For dependent current sources, the procedure is modified as follow:

1. Step 1 and 2 are the same as those applied for independent sources.
2. Step 3 is modified as follows: The procedure is essentially the same as that applied for dependent current sources, except now the dependent sources have to be defined in terms of the chosen mesh currents to ensure that the final equations have only mesh currents as the unknown quantities.
3. Step 4 is as before.
Ex. 17-8 Write the mesh currents for the network in Fig. 17.14 having an
dependent current source.

Steps 1 and 2 are defined in Fig. 17.14.
Step 3: \[ E_n - I_1Z_1 + I_2Z_2 - I_1Z_1 = 0 \]
and \[ kI = I_1 - I_2 \]
Now \[ I_1 = I_2 \], so that \[ kI = I_1 - I_2 \], or \[ I_2 = I_1(1-k) \]
The result is two equations and two unknowns.

Ex. 17-12 Determine the voltage across the inductor for the network in Fig. 17.23.

Steps 1 and 2 are as indicated in Fig. 17.24,

Steps 3: Note Fig. 17.25 for the application of KCL to node \( V_1 \):
\[
\sum I_1 = \sum I_2
\]
\[
0 = I_1 + I_2 + I_3
\]
\[
V_1 - E + V_1' = V_1 - V_2 = 0
\]

Figure 17.25 Applying KCL to the nodes \( V_1 \) in Fig. 17.24.

Figure 17.26 Applying KCL to the nodes \( V_2 \) in Fig. 17.24.

Rearranging terms:
\[
V' \left[ \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right] - V \left[ \frac{1}{Z_2} + \frac{1}{Z_3} \right] = E
\]
Note Fig. 17.26 for the application of KCL to the node \( V_2 \).
\[
0 = I_1 + I_2 + I_3
\]
\[
V_1' - V_2 = V_2 - Z_2^{-1} Z_3^{-1} = 0
\]

Grouping equations:
\[
V_1' \left[ \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right] - V_2 \left[ \frac{1}{Z_1} + \frac{1}{Z_3} \right] = \frac{E}{Z_1}
\]
\[
V_1' \left[ \frac{1}{Z_1} + \frac{1}{Z_3} \right] - V_2 \left[ \frac{1}{Z_1} + \frac{1}{Z_3} \right] = I
\]
\[
\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} = 0.5\Omega + \frac{1}{10\Omega} + \frac{1}{2\Omega} = 2.5\Omega \angle -2.29^\circ
\]
\[
\frac{1}{Z_1} + \frac{1}{Z_3} = 0.5\Omega + \frac{1}{2\Omega} = 0.539mS \leq 21.80^\circ
\]
**Dependent Current Sources**

For dependent current sources, the procedure is modified as follow:

1. **Step 1 and 2 are the same as those applied for independent sources.**
2. **Step 3 is modified as follows:** Treat each dependent current source like an independent source when KCL is applied to each defined node.
3. **Step 4 is as before.**

---

**Ex. 17-13** Write the nodal equations for the network in Fig. 17.28 having a dependent current source.

**Figure 17.28** Applying nodal analysis to a network with a current-controlled current source.

---

**Ex. 17-14** Write the nodal equations for the network in Fig. 17.29 having an independent source between two assigned nodes.

**Figure 17.29** Applying nodal analysis to a network with an independent voltage source between defined nodes.

---

**Dependent Voltage Sources between Defined Nodes**

For dependent voltage sources between defined nodes, the procedure is modified as follow:

1. **Step 1 and 2 are the same as those applied for independent voltage sources.**
2. **Step 3 is modified as follows:** The procedure is essentially the same as that applied for independent voltage sources, except now the dependent sources have to be defined in terms of the chosen voltages to ensure that the final equations have only nodal voltages as their unknown quantities.
3. **Step 4 is as before.**

---

**Ex. 17-15** Write the nodal equations for the network in Fig. 17.30 having a dependent voltage source between two defined nodes.

**Figure 17.30** Applying nodal analysis to a network with a voltage-controlled voltage source.

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**Homework** 17: 2-4, 5, 6, 14, 15

---

**HW 17-10** An electrical system is rated 10 kVA, 200V at a leading power factor.

- a. Determine the impedance of the system in rectangular coordinates.
- b. Find the average power delivered to the system.
Network Theorems (AC) - Introduction

This module will deal with network theorems of ac circuit rather than dc circuits previously discussed. Due to the need for developing confidence in the application of the various theorems to networks with controlled (dependent) sources include independent sources and dependent sources. Theorems to be considered in detail include the superposition theorem, Thevinin’s theorem, maximum power transform theorem.

Superposition Theorem

The superposition theorem eliminated the need for solving simultaneous linear equations by considering the effects of each source independently in previous module with dc circuits. To consider the effects of each source, we had to remove the remaining sources. This was accomplished by setting voltage sources to zero (short-circuit representation) and current sources to zero (open-circuit representation). The current through, or voltage across, a portion of the network produced by each source was then added algebraically to find the total solution for the current or voltage.

The only variation in applying this method to ac networks with independent sources is that we are now working with impedances and phasors instead of just resistors and real numbers.

Ex. 18-1 Using the superposition theorem, find the current \( I \) through the 4Ω resistance \((X_L2)\) in Fig. 18.1.

For the redrawn circuit (Fig.18.2), we have:

\[
\begin{align*}
Z_{e1} &= jX_{e1} = j4\Omega \\
Z_2 &= jX_2 = j4\Omega \\
Z_1 &= -jX_1 = -j3\Omega
\end{align*}
\]

Considering the effects of the voltage source \( E_1 \) (Fig.18.3), we have:

\[
\begin{align*}
Z_{e2} &= Z_{e1} + Z_1 + Z_2 = \frac{-j4\Omega - j3\Omega}{10\Omega} = \frac{-10\Omega}{10\Omega} = -1 + j0 = -1\angle 0^\circ \\
l_1 &= \frac{E_1}{Z_{e2} + Z_2} = \frac{10\Omega}{-1\angle 0^\circ + j4\Omega} = \frac{10\Omega}{-1\angle 0^\circ + j4\Omega} = 1.25\angle -90^\circ.
\end{align*}
\]
Figure 18.3 Determining the effect of the voltage source $E_1$ on the current $I$ of the network in Fig. 18.1.

\[ I' = \frac{Z_1 I}{Z_1 + Z_3} \quad \text{(current divider rule)} \]

Consider the effects of the voltage source (Fig. 18.8). Applying Ohm’s law gives us

\[ I' = \frac{E_1}{Z_1 + Z_3} = \frac{200\angle 0^\circ}{6.32\Omega - 18.43^\circ} = 3.16\angle 48.43^\circ \]

The total current through the 6Ω resistor (Fig. 18.10) is

\[ I = I' + I'' = 1.94\angle 108.43^\circ + 3.16\angle 48.43^\circ \]

\[ = (-0.60\angle 180.4^\circ) + (2.10\angle 2.46^\circ) \]

\[ = 1.54\angle 4.42^\circ \]

Figure 18.4 Determining the effect of the voltage source $E_2$ on the current $I$ of the network in Fig. 18.1.

\[ \frac{75.3}{75.3} + j30 = 25.1 + j30 \]

Figure 18.5 Determining the resultant current for the network in Fig. 18.1.

\[ I = \frac{I'}{2} = 2.54\angle 90^\circ \]

Ex. 18.2 Using the superposition, find the current $I$ through the 6Ω resistor in Fig. 18.6.

\[ E_2 = 20V \angle 30^\circ \]

Figure 18.6 Example 18.2.

Ex. 18.3 Using the superposition, find the voltage across the 6Ω resistor in Fig. 18.6. Check the results against $V_{6\Omega} = I(6\Omega)$, where $I$ is the current found through the 6Ω resistor in Example 18.2.

\[ V_{E1} = I'(6\Omega) = (1.94\angle 108.43^\circ)(6\Omega) = 11.4\angle 108.43^\circ \]

For the voltage source,

\[ V_{E2} = I''(6\Omega) = (3.16\angle 48.43^\circ)(6\Omega) = 18.96\angle 48.43^\circ \]

Check the result, we have

\[ V_{6\Omega} = I(6\Omega) = (4.42\angle 70.2^\circ)(6\Omega) = 26.5V\angle 70.2^\circ \quad (checks) \]
Dependent Sources  For dependent sources in which the controlling variable is not determined by the network to which the superposition is to be applied, the application of the theorem is basically the same as for independent sources.

Ex. 18-5  Using the superposition, determine the current $I_2$ for the network in Fig. 18.18. The quantities $\mu$ and $h$ are constants.

**Figure 18.18**  Example 18.5.

![Image](image1.png)

With a portion of the system (Fig. 18.19),

$Z_1 = R_1 = 4\Omega$

$Z_2 = R_2 + jX_2 = 6\Omega + j8\Omega$

For the voltage source (Fig. 18.20),

$I'' = \frac{\mu V}{Z_1 + Z_2} = \frac{\mu V}{4\Omega + 6\Omega + j8\Omega} = \frac{\mu V}{10\Omega + j8\Omega} = \frac{\mu V}{12.8\Omega / 38.66^\circ} = 0.078\frac{\mu V}{\Omega} - 38.66^\circ$

**Figure 18.19**  Assigning the subscripted impedances to the network in Fig. 18.18.

![Image](image2.png)

**Figure 18.20**  Determining the effect of the voltage-controlled voltage source on the current $I_2$ for the network in Fig. 18.18.

For the current source (Fig. 18.21),

$I'' = Z_1(hI) = \frac{(4\Omega)(hI)}{12.8\Omega / 38.66^\circ} = 4(0.078)hI / -38.66^\circ = 0.312hI / -38.66^\circ$

For the current $I_1$ is

$I_1 = I' + I'' = 0.078\mu V / \Omega - 38.66^\circ + 0.312hI - 38.66^\circ$

For $V = 10V / 0^\circ$, $\mu = 20$, and $h = 100$,

$I_2 = 0.078(20)(10V / 0^\circ) / \Omega / -38.66^\circ + 0.312(100)(20mA / 0^\circ) / \Omega / -38.66^\circ$

$= 15.60A / -38.66^\circ + 0.62A / -38.66^\circ = 16.22A / -38.66^\circ$

**Figure 18.21**  Determining the effect of the current-controlled current source on the current $I_2$ for the network in Fig. 18.18.

Thevenin’s Theorem

Thevenin’s theorem, as stated for sinusoidal ac circuits, is changed only to include the term impedance instead of resistance, that is,

any two-terminal linear ac network can be replaced with an equivalent circuit consisting of a voltage source and an impedance in series, as shown in Fig. 18.23.

Since the reactances of a circuit are frequency dependent, the Thevenin circuit found for a particular network is applicable only at one frequency. The steps required to apply this method to dc circuits are repeated here with changes for sinusoidal ac circuits. As before, the only change is the replacement of the term resistance with impedance. Again, dependent and independent sources are treated separately.

**Figure 18.22**  Thevenin equivalent circuit for ac networks.

![Image](image3.png)

**Figure 18.23**  Determining the effect of the voltage-controlled voltage source on the network in Fig. 18.24.

**Figure 18.24**  Example 18.7.

Steps 1 and 2 (Fig. 18.25):

$Z_i = jX_1 = j8\Omega$

$Z_i = -jX_1 = -j8\Omega$

Step 3 (Fig. 18.26):

$Z_0 = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(6j\Omega - 16j\Omega)}{(6j\Omega + 16j\Omega)} = 2.67\Omega / -90^\circ$

Step 4 (Fig. 18.27):

$E_0 = Z_0 E = 2.67\Omega (12\Omega) = -50V / -90^\circ$

Ex. 18-7  Find the Thevenin equivalent circuit for the network external to resistor $R$ in Fig. 18.24.

**Figure 18.25**  Assigning the subscripted impedances to the network in Fig. 18.24.

![Image](image4.png)

**Figure 18.26**  Determining the Thevenin impedance for the network in Fig. 18.24.

**Figure 18.27**  Determining the open-circuit Thevenin voltage for the network in Fig. 18.24.

**Figure 18.28**  Example 18.7.
Step 5: The Thevenin equivalent circuit is shown in Fig. 18.28.

Ex. 18-8 Find the Thevenin equivalent circuit for the network external to resistor to branch a-a’ in Fig. 18.24.

Dependent Sources For dependent sources with a controlling variable not in the network under investigation, the procedure indicated above can be applied. However, for dependent sources of the other type, where the controlling variable is part of the network to which the theorem is to be applied, another approach must be used. The new approach to Thevenin’s theorem can best be introduced at this stage in the development by considering the Thevenin equivalent circuit in Fig. 18.39(a). As indicated in fig. 18.39(b), the open-circuit terminal voltage (E_{oc}) of the Thevenin equivalent circuit is the Thevenin equivalent voltage; that is

\[ E_{oc} = E_{Th} \]

If the external terminals are short circuited as in Fig. 18.39(c), it resulting short-circuit current is determined by

\[ I_{sc} = \frac{E_{oc}}{Z_{Th}} \]

or, rearranged,

\[ Z_{Th} = E_{oc} / I_{sc} \]

Ex. 18-11 Determine the Thevenin equivalent circuit for the network in Fig. 18.24.

From Fig. 18.47, \( E_{in} \) is

\[ E_{in} = E_{oc} = -hI \left( R_{in} \right) / R_{1} = -hR_{in}I \]

Method 1: See Fig. 18.48. \( Z_{Th} = R_{in} / R_{1} - jX_{c} \)

Method 2: See Fig. 18.49. \( I_{sc} = \left( R_{in} / R_{1} \right) \left( R_{1} / R_{1} + M \right) \)

and \( Z_{Th} = \frac{E_{oc}}{I_{sc}} = \frac{E_{oc}}{I_{sc}} = \frac{E_{oc}}{I_{sc}} = \frac{E_{oc}}{I_{sc}} \)

Method 3: See Fig. 18.50. \( I_{sc} = \frac{E_{oc}}{R_{in} / R_{1} - jX_{c}} \)

and \( Z_{Th} = \frac{E_{oc}}{I_{sc}} = R_{in} / R_{1} - jX_{c} \)
Maximum Power Transfer Theorem

When applied to ac circuits, the maximum power transfer theorem states that maximum power will be delivered to a load when the load impedance is the conjugate of the Thevenin impedance across its terminals.

That is, for Fig. 18.81, for maximum power transfer to the load,

$$Z_L = Z_{Th}$$ and $$\theta_L = -\theta_{Th}$$

or, in rectangular form,

$$R_L = R_{Th}$$ and $$\pm jX_{load} = \mp jX_{Th}$$

Figure 18.81 Defining the conditions for maximum power transfer to a load.

Ex. 18-19 Find the load impedance in Fig. 18.83 for maximum power to the load, and find the maximum power.

$$Z_L = 13.33\Omega - 36.87\Omega = 10.66\Omega - j8\Omega$$

To find the maximum power, we must find

$$E_{Th} = \frac{Z_E}{Z_L + Z_L}$$

$$= \frac{8\Omega - 90\Omega}{8\Omega - 90\Omega} = 72\Omega - 90\Omega = 12\Omega - 90\Omega$$

Then

$$P_{max} = \frac{E_{Th}^2}{4R}$$

$$= \frac{144}{42.64} = 3.38\ W$$

Figure 18.84 Determining (a) $$Z_{Th}$$ and (b) $$E_{Th}$$ for the network external to the load in Fig. 18.83.

The conditions just mentioned will make the total impedance of the circuit appear purely resistive, as indicated in Fig. 18.82:

$$Z_L = (R + jX) + (R - jX)$$

and $$Z_{Th} = 2R$$

Since the circuit is purely resistive, the power factor of the circuit under maximum power conditions is 1; that is, $$F_p = 1$$ (maximum power transfer)

The magnitude of the current $$I$$ in Fig. 18.82 is

$$I = \frac{E_{Th}}{Z_{Th}} = \frac{E_{Th}}{2R}$$

The maximum power to the load is

$$P_{max} = I^2 R = \left(\frac{E_{Th}}{2R}\right)^2 R$$

and

$$P_{max} = \frac{E_{Th}^2}{4R}$$

Figure 18.82 Conditions for maximum power transfer to Z_L.

HW 18-6 Using superposition, determine the current $$I_L$$ ($$h = 100$$) for the network in Fig. 18.112.

$$Z_L = 20 \Omega \angle 0^\circ$$

$$Z_{Th} = 10 \Omega \angle 90^\circ$$

$$I = 2 \ mA \angle 0^\circ$$

$$E = 10 \ V \angle 0^\circ$$

$$\Gamma = \frac{Z_L}{Z_{Th}} = \frac{(20 \Omega \angle 0^\circ)(10 \Omega \angle -90^\circ)}{20 \Omega + j10 \Omega} = 0.447 \ mA \angle -26.57^\circ$$

$$I_L = \Gamma + \Gamma'$$ (Node's theorem)

$$= 179 \ mA \angle -26.57^\circ - 0.447 \ mA \angle -26.57^\circ$$

$$= 178.55 \ mA \angle -26.57^\circ$$

Figure 18.112 Problems 6 and 20.

Homework 18: 1, 2, 6, 12, 13, 19, 39
Power (AC) - Introduction

The discussion of power in earlier module of response of basic elements included only the average or real power delivered to an ac network. We now examine the total power equation in a slightly different form and introduce two additional types of power: apparent and reactive.

General Equation

For any system as in Fig. 19.1, the power delivered to a load at any instant is defined by the product of the applied voltage and the resulting current; that is,

\[ p = vi \]

In this case, since \( v \) and \( i \) are sinusoidal quantities, let us establish a general case where

\[ v = V_e \sin(\omega t + \phi) \]

and

\[ i = I_e \sin(\alpha t) \]

\[ \phi = \alpha \]

\[ p = V_e I_e \sin(\phi) \]

\[ \cos(\phi) \]

**Key Words:** Power, Apparent Power, Power Factor, Power Meter, Effective Resistance

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Sunghoon Jang
The chosen \( v \) and \( i \) include all possibilities because, if the load is purely resistive, \( \theta = 0^\circ \). If the load is purely inductive or capacitive, \( \theta = 90^\circ \) or \( \theta = -90^\circ \), respectively.

Substituting the above equations for \( v \) and \( i \) into the power equation results in:

\[
p = Vf \cos \omega t (1 - \cos 2\omega t) + Vf \sin \omega t \sin 2\omega t
\]

If we now apply a number of trigonometric identities, the following form for the power equation results in:

\[
p = VI \cos \theta (1 - \cos 2\omega t) + VI \sin \theta \sin 2\omega t
\]

where \( V \) and \( I \) are rms values.

### Resistive Circuit

For a purely resistive circuit (such as that in Fig. 19.2), \( v \) and \( i \) are in phase, and \( \theta = 0^\circ \), as appearing in Fig. 19.3. Substituting \( \theta = 0^\circ \) into the above equation.

\[
p = VI \cos \theta (1 - \cos 2\omega t) + VI \sin \theta \sin 2\omega t
\]

The chosen \( v \) and \( i \) include all possibilities because, if the load is purely resistive, \( \theta = 0^\circ \). If the load is purely inductive or capacitive, \( \theta = 90^\circ \) or \( \theta = -90^\circ \), respectively.

Consider also that since the peak and average values of the power curve are the same, the curve is always above the horizontal axis. This indicates that the total power delivered to a resistor will be dissipated in the form of heat.

For a purely resistive circuit (such as that in Fig. 19.2), \( v \) and \( i \) are in phase, and \( \theta = 0^\circ \), as appearing in Fig. 19.3. Substituting \( \theta = 0^\circ \) into the above equation.

\[
p = VI \cos \theta (1 - \cos 2\omega t) + VI \sin \theta \sin 2\omega t
\]

The energy dissipated by the resistor (WR) over one full cycle of the applied voltage is the area under the power curve in Fig. 19.3. It can be found using the following equation:

\[
W = pt
\]

where \( p \) is the average value and \( t \) is the period of the applied voltage, that is, the total power delivered to a resistor will be dissipated in the form of heat.

The average (real) power is VI; or, as a summary,

\[
p = VI \cos \theta (1 - \cos 2\omega t) + VI \sin \theta \sin 2\omega t
\]

Consider also that since the peak and average values of the power curve are the same, the curve is always above the horizontal axis. This indicates that the total power delivered to a resistor will be dissipated in the form of heat.

For a purely resistive circuit (such as that in Fig. 19.2), \( v \) and \( i \) are in phase, and \( \theta = 0^\circ \), as appearing in Fig. 19.3. Substituting \( \theta = 0^\circ \) into the above equation.

\[
p = VI \cos \theta (1 - \cos 2\omega t) + VI \sin \theta \sin 2\omega t
\]

The energy dissipated by the resistor (WR) over one full cycle of the applied voltage is the area under the power curve in Fig. 19.3. It can be found using the following equation:

\[
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\]

where \( p \) is the average value and \( t \) is the period of the applied voltage, that is, the total power delivered to a resistor will be dissipated in the form of heat.

The average (real) power is VI; or, as a summary,

\[
p = VI \cos \theta (1 - \cos 2\omega t) + VI \sin \theta \sin 2\omega t
\]

Consider also that since the peak and average values of the power curve are the same, the curve is always above the horizontal axis. This indicates that the total power delivered to a resistor will be dissipated in the form of heat.

For a purely resistive circuit (such as that in Fig. 19.2), \( v \) and \( i \) are in phase, and \( \theta = 0^\circ \), as appearing in Fig. 19.3. Substituting \( \theta = 0^\circ \) into the above equation.

\[
p = VI \cos \theta (1 - \cos 2\omega t) + VI \sin \theta \sin 2\omega t
\]

The energy dissipated by the resistor (WR) over one full cycle of the applied voltage is the area under the power curve in Fig. 19.3. It can be found using the following equation:

\[
W = pt
\]

where \( p \) is the average value and \( t \) is the period of the applied voltage, that is, the total power delivered to a resistor will be dissipated in the form of heat.

The average (real) power is VI; or, as a summary,

\[
p = VI \cos \theta (1 - \cos 2\omega t) + VI \sin \theta \sin 2\omega t
\]

Consider also that since the peak and average values of the power curve are the same, the curve is always above the horizontal axis. This indicates that the total power delivered to a resistor will be dissipated in the form of heat.

For a purely resistive circuit (such as that in Fig. 19.2), \( v \) and \( i \) are in phase, and \( \theta = 0^\circ \), as appearing in Fig. 19.3. Substituting \( \theta = 0^\circ \) into the above equation.

\[
p = VI \cos \theta (1 - \cos 2\omega t) + VI \sin \theta \sin 2\omega t
\]

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\]

where \( p \) is the average value and \( t \) is the period of the applied voltage, that is, the total power delivered to a resistor will be dissipated in the form of heat.

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\[
W = pt
\]

where \( p \) is the average value and \( t \) is the period of the applied voltage, that is, the total power delivered to a resistor will be dissipated in the form of heat.

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\]

Consider also that since the peak and average values of the power curve are the same, the curve is always above the horizontal axis. This indicates that the total power delivered to a resistor will be dissipated in the form of heat.
Its magnitude is determined by
\[ S = VI \] (volt-ampere, VA)
And, since
\[ V =IZ \text{ and } I = \frac{V}{Z} \]
then
\[ S = \frac{V^2}{Z} \] (VA)
and
\[ S = \frac{V^2}{Z} \] (VA)

The average power to the load in Fig. 19.4 is
\[ P = \frac{VI}{\cos \theta} \] (W)
and the power factor of a system \( F_p \) is
\[ F_p = \cos \theta = \frac{P}{S} \] (unitless)

The reason for rating some electrical equipment in kilovolt-amperes rather than in kilowatts can be described using the configuration in Fig. 19.7. The load has an apparent power rating of 10 kVA and a current demand of 70 A is above the rated value and could damage the load element, yet the reading on the wattmeter is relatively low since the load is highly reactive.

Note that over one full cycle of \( p_L \) (T2), the area above the horizontal axis in Fig. 19.9 is exactly equal to that below the axis. This indicates that over a full cycle of \( p_L \), the power delivered by the sources to the inductor is exactly equal to that returned to the source by the inductor.

The net flow of power to the pure (ideal) inductor is zero over a full cycle, and no energy is lost in the transaction.

The power absorbed or returned by the inductor at any instant of time \( t \) can be found simply by substituting \( t \) into Eq. (19.11). The peak value of the curve \( VI \) is defined as the reactive power associated with a pure inductor. The symbol for reactive power is \( Q \), and its unit of measure is the volt-ampere reactive (VAR).

\[ Q_L = VI \sin \theta \] (volt-ampere reactive, VAR)
where \( \theta \) is the phase angle between \( V \) and \( I \).

For the inductor,
\[ Q_L = VI \] (VAR) \hspace{1cm} (19.13)

or, since \( V = IX_L \) or \( I = V / X_L \)
\[ Q_L = I^2 X_L \] (VAR) \hspace{1cm} or \hspace{1cm} \[ Q_L = \frac{V^2}{X_L} \] (VAR)

The energy stored by the inductor during the positive portion of the cycle (Fig. 19.9) is equal to that returned during the negative portion and can be determined using the following equation:
\[ W = Pt \]
Where \( P \) is the average value for the interval and \( t \) is the associated interval of time.

The average value of the positive portion of a sinusoid equals 2(peak value/frequency, where \( f \) is the frequency of the power curve, we have
\[ W_L = \frac{V^2}{f_1} \] (J) \hspace{1cm} (19.17)

For the inductive circuit in Fig. 19.10,

a. Find the instantaneous power level for the inductor at times \( t_1 \) through \( t_5 \).

b. Plot the results of part (a) for one full period of the applied voltage.

c. Find the average value of the curve of part (b) over one full cycle of the applied voltage and compare the peak value of each pulse with the value determined by Eq. (19.13).

d. Find the energy stored or released for any one pulse of the power curve.
b. The resulting plot of $V_L$ and $i_L$ appears in Fig. 19.11.

c. The average value for the curve in Fig. 19.11 is 0W over one full cycle of the applied voltage. The peak value of the curve is 10W which compares directly with that obtained from the product $V_i V_o$ of the applied voltage. The peak value of the curve is 10W which compares directly with that obtained from the product $V_i V_o$

$$V = V_i V_o$$
$$P = 10W$$

The process of introducing reactive elements to bring the power-factor closer to unity is called power-factor correction. Since most loads are inductive, the process normally involves introducing elements with capacitive terminal characteristics having the sole purpose of improving the power factor.

Power-Meter

The power meter in Fig. 19.34 uses a sophisticated electronic package to sense the voltage and current levels and has an analog-to-digital conversion unit that display the levels in digital form. It is capable of providing a digital readout for distorted nonsinusoidal waveforms, and it can provide the phase power, total power, apparent power, reactive power, and power factor. The power quality analyzer in Fig. 19.35 can also display the real, reactive, and apparent power levels along with the power factor. However, it has a board range of other options, including providing the harmonic content of up to 51 terms for the voltage, current, and power.

Effective Resistance

The resistance of a conductor as determined by the equation $R = \rho l / A$ is often called the ohmic, or geometric resistance. It is a constant quantity determined only by the material used and its physical dimensions. In ac circuits, the actual resistance of a conductor (called effective resistance) differs from the dc resistance because of the varying currents and voltages that introduce effects not present in dc circuits. These effects include radiation losses, skin effect, eddy currents, and hysteresis losses.
Effective Resistance – Experimental Procedure

The effective resistance of an ac circuit cannot be measured by the ratio $V/I$ since this ratio is now the impedance of a circuit that may have both resistance and reactance. The effective resistance can be found, however, by using the power equation $P = I^2R$, where

$$R_{\text{eff}} = \frac{P}{I^2}$$

A wattmeter and ammeter are therefore necessary for measuring the effective resistance of an ac circuit.

Effective Resistance – Radiation Losses

The radiation loss is the loss of energy in the form of electromagnetic waves during the transfer of energy in the from one element to another. This loss in energy requires that the input power be larger to establish the same current $I$, causing $R$ to increases as determined by Eq. (19.31). At a frequency of 60Hz, the effects of radiation losses can be completely ignored. However, at radio frequencies, this is important effect and may in fact become the main effect in an electromagnetic device such as an antenna.

Effective Resistance – Skin Effect

The explanation of skin effect requires the use of some basic concepts previously described. A magnetic field exist around every current-carrying conductor. Since the amount of charge flowing in ac circuits changes with time, the magnetic field surrounding the moving charge (current) also changes. Recall also that a wire placed in a changing magnetic field will have an induced voltage across its terminals as determined by Faraday’s law, $e = -N \times \Phi/(dt)$. The higher the frequency of the changing flux as determined by an alternating current, the greater the induced voltage.

Effective Resistance – Hysteresis and Eddy current losses

As mentioned earlier, hysteresis and eddy current losses appear when a ferromagnetic material is placed in the region of a changing magnetic field. To describe eddy current losses in greater detail, we consider the effects of an alternating current passing through coil wrapped around a ferromagnetic core. As the alternating current passes through the coil, it develops a changing magnetic flux $\Phi$ linking both coil and the core that develops an induced voltage and geometric resistance of the core $\rho / (l/A)$ cause currents to be developed within the core, called eddy currents.

The currents flow in circular paths, as shown in fig. 19.37, changing direction with the applied ac potential. The eddy current losses are determined by

$$P_{\text{eddy}} = \frac{i_{\text{eddy}}^2}{R_{\text{core}}}$$

The eddy current loss is proportional to the square of the frequency times the square of magnetic field strength:

$$P_{\text{eddy}} \propto f^2B^2$$

Eddy current losses can be reduced if the core is constructed of thin, laminated sheets of ferromagnetic material insulated from one another and aligned parallel to the magnetic flux.

In terms of the frequency of the applied signal and the magnetic field strength produced, the hysteresis loss is proportional to the frequency to the 1st power times the magnetic field strength to the nth power:

$$P_{\text{hyst}} \propto f^1B^n$$

Where $n$ can vary from 1.4 to 2.6, depending on the material under consideration.

Hysteresis losses can be effectively reduced by the injection of small amounts of silicon into the magnetic core, constituting some 2% or 3% of the total composition of the core.

Homework 19: 2-6, 10-13, 16-18

HW 19-10 An electrical system is rated 10 kVA, 200V at a leading power factor.

a. Determine the impedance of the system in rectangular coordinates.

b. Find the average power delivered to the system.

a. $I_T = \frac{S_T}{E} = 10,000 \text{ VA} / 200 \text{ V} = 50 \text{ A}$

$0.5 \Rightarrow 60^\circ$ leading

$\therefore I_2$ leads $E$ by $60^\circ$

$Z_T = \frac{E}{I_2} = \frac{200 \text{ V} \angle 0^\circ}{50 \text{ A} \angle 60^\circ} = 4 \Omega \angle -60^\circ = 2 \Omega - j 0.464 \Omega = R - j X_C$

b. $P_T = \frac{S_T^2}{2T} \Rightarrow P_T = P_T = 0.5(10,000 \text{ VA}) = 5000 \text{ W}$

Homework 19: 2-6, 10-13, 16-18
Resonance - Introduction

The resonance circuit is a combination of R, L, and C elements having a frequency response characteristic similar to the one appearing in Fig. 20.1. Note in the figure that the response is a maximum for the frequency \( f_r \), decreasing to the right and left of the frequency. In other words, for a particular range of frequencies, the response will be near or equal to the maximum. When the response is at or near the maximum, the circuit is said to be in a state of resonance.

Key Words: Series Resonance, Total Impedance, Quality Factor, Selectivity

Resonance - Series Resonance Circuit

A resonant circuit must have an inductive and a capacitive element. A resistive element is always present due to the internal resistance \( R_s \), the internal resistance of the response curve \( R_{\text{eq}} \). The basic configuration for the series resonant circuit appears in Fig. 20.2(a) with the resistive elements listed above. The "cleaner" appearance in Fig. 20.2(b) is a result of combining the series resistive elements into one total value. That is \[ R = R_s + R_L + R_d \]
The total impedance of this network at any frequency is determined by

\[ Z = R + jX_L - jX_C = R + j(X_L - X_C) \]

The resonant frequency can be determined in terms of the inductance and capacitance by examining the defining equation for resonance [Eq. (20.2)]:

\[ X_L = X_C \]  

The current through the circuit at resonance is

Substituting \( \omega \) yields \( \omega L = \frac{1}{\omega C} \)

and \( \omega = \frac{1}{\sqrt{LC}} \)

or \( f_r = \frac{1}{2\pi\sqrt{LC}} \)

\( f = \text{hertz (Hz)}, L = \text{henries (H)}, C = \text{farads (F)} \)

The resonant conditions described in the introduction occur when

\[ L = X_C \]

which is the maximum current for the circuit in Fig. 20.2 for an applied voltage \( E \) since \( Z_T \) is a minimum value. Consider also that the input voltage and current are in phase at resonance.

The resonant conditions in the introduction occur when

\[ L = X_C \]

And, since \( X_L = X_C \), the magnitude of \( V_L \) equals \( V_C \) at resonance; that is,

\[ V_L = V_C \]

Substituting for an inductive reactance in Eq. (20.8) at resonance gives

\[ Q_s = X_L / R \]

If the resistor \( R \) is just the reactance of the coil \( R_L \), we can speak of the \( Q \) of the coil, where

\[ Q_{coil} = Q_L = X_L / R_L \]

The \textit{average power} to the resistor at resonance is equal to \( I^2R \), and the \textit{reactive power} to the capacitor and inductor are \( P \) and \( Q \) respectively. The power triangle at resonance (Fig. 20.4) shows that the total \textit{apparent power} is equal to the average power dissipated by the resistor since \( Q_{s} = Q_{C} \). The \textit{power factor} of the circuit at resonance is

\[ F_p = \cos \theta = P/S \] and \[ F_{ps} = 1 \]

\[ \text{Series Resonance - Quality Factor (Q)} \]

The quality factor \( Q \) of a series resonant circuit is defined as the ratio of the reactive power of either the inductor or the capacitor to the average power of the resistor at resonance; that is,

\[ Q_s = \frac{\text{reactive power}}{\text{average power}} \]

The quality factor is also an indication of how much energy is placed in storage compared to that dissipated. The lower the level of dissipation for the same reactive power, the larger the \( Q \) factor and the more concentrated and intense the region of resonance.

\[ Q_s = \frac{\text{reactive power}}{\text{average power}} \]

\[ Q_s = \frac{\text{reactive power}}{\text{average power}} \]

\[ Q_s = \frac{\text{reactive power}}{\text{average power}} \]
The total impedance-versus-frequency curve for the series resonant circuit in Fig. 20.2 can be found by applying the impedance-versus-frequency curve for each element of the equation just derived, written in the following form:

\[ Z(f) = \sqrt{(R(f))^2 + (X_L(f) - X_C(f))^2} \]

Where \( Z(f) \) “means” the total impedance as a function of frequency. For the frequency range of interest, we assume that the resistance \( R \) does not change with frequency, resulting in the plot in Fig. 20.8.

If we place Figs. 20.9 and 20.10 on the same set of axes, we obtain the curves in Fig. 20.11. The condition of resonance is now clearly defined by the point of intersection, where \( X_L = X_C \). For frequency less than \( f_s \), it is also quite clear that the network is primarily capacitive \((X_L > X_C)\). For frequencies above the resonant condition, \( X_L > X_C \), and network is inductive.

At low frequencies, \( \theta \) approaches \(-90^\circ\) (capacitive), as shown in Fig. 20.13. At high frequencies, \( X_L > X_C \) and \( \theta \) approaches \(90^\circ\). In general, therefore, for a series resonant circuit:

- \( f < f_s \): network capacitive, \( I \) leads \( E \)
- \( f = f_s \): network capacitive, \( I \) and \( E \) are in phase
- \( f > f_s \): network capacitive, \( E \) leads \( I \)

The curve for the inductance, as determined by the reactance equation, is a straight line intersecting the origin with a slope equal to the inductance of the coil. The mathematical expression for any straight line in a two-dimensional plane is given by

\[ y = mx + b \]

Thus, for the coil,

\[ X_L = 2\pi f L + 0 = (2\pi f L)(f) + 0 \]

(\( 2\pi L \) is the slope), producing the results shown in Fig. 20.9.

For the capacitor, which becomes \( X_C = \frac{1}{2\pi f C} \) or \( X_C f = \frac{1}{2\pi C} \) (where \( 2\pi \) is the variable), the equation for a hyperbola, where

\[ y \ (\text{variable}) = \frac{1}{2\pi C} \]

The hyperbolic curve for \( X_C(f) \) is plotted in Fig. 20.10. In particular, note its very large magnitude at low frequencies and its rapid drop off as the frequency increases.

There is a definite range of frequencies at which the current is near its maximum value and the impedance is at a minimum. Those frequencies corresponding to 0.707 of maximum current are called the band frequencies, cutoff frequencies, half-power frequencies, or corner frequencies. They are indicated by \( f_1 \) and \( f_2 \) in Fig. 20.14. The range of frequencies between the two is referred to as bandwidth (BW) of the resonant circuit.

Half-power frequencies are those frequencies at which the power delivered is one-half that delivered at the resonant frequency; that is

\[ P_{\text{HPF}} = \frac{1}{2} P_{\text{max}} \]

where

\[ P_{\text{max}} = I_{\text{max}}^2 R \]
Since the resonant circuit is adjusted to select a band of frequencies, the curve in Fig. 20.14 is called the selective curve. The term is derived from the fact that on must be selective in choosing the frequency to ensure that in the bandwidth. The smaller bandwidth, the higher the selectivity. The shape of the curve, as shown in Fig. 20.15, depends on each element of the series R-L-C circuit. If resistance is made smaller with a fixed inductance and capacitance, the bandwidth decreases and the selectivity increases.

The bandwidth (BW) is

\[ BW = \frac{f_l}{Q_s} \]

It can be shown through mathematical manipulations of the pertinent equations that the resonant frequency is related to the geometric mean of the band frequencies; that is

\[ f_r = \sqrt{f_1 f_2} \]

**Ex. 20-1**

a. For the series resonant circuit in Fig. 20.19, find I, \( V_R \), \( V_L \), and \( V_C \) at resonance.

b. What is the Q of the circuit?

c. If the resonant frequency is 5000 Hz, find the bandwidth.

d. What is the power dissipated in the circuit at the half-power frequencies?

\[ a. \quad Z_f = R = 2\Omega \\
I = \frac{E}{Z_f} = \frac{10V / 0^\circ}{2\Omega} = 5A / 0^\circ \\
V_R = E = 10V / 0^\circ \\
V_L = (1\Omega)(X_f) / 90^\circ = (5A / 0^\circ)(10\Omega / 90^\circ) = 50V / 90^\circ \\
V_C = (1\Omega)(X_f) / -90^\circ = (5A / 0^\circ)(10\Omega / -90^\circ) = 50V / -90^\circ \\
\]

\[ b. \quad Q_s = \frac{X_f}{R} = \frac{10\Omega}{2\Omega} = 5 \\
\]

\[ c. \quad BW = f_2 - f_1 = f_r / 5 = 1000 \text{ Hz} \\
\]

\[ d. \quad P_{HPF} = \frac{1}{2}P_{avg} = \frac{1}{2}f_r R = \left(\frac{1}{2}\right)(5A)^2(2\Omega) = 25 \text{ W} \]

**Ex. 20-2**

The bandwidth of a series resonant circuit is 400 Hz.

a. If the resonant frequency is 4000 Hz, what is the value of \( Q_s \)?

b. If \( R = 10\Omega \), what is the value of \( X_f \) at resonance?

c. Find the inductance \( L \) and capacitance \( C \) of the circuit.

\[ a. \quad BW = \frac{f_2}{Q_s} \quad \text{or} \quad Q_s = \frac{f_2}{BW} = \frac{4000 \text{ Hz}}{400 \text{ Hz}} = 10 \\
\]

\[ b. \quad Q_s = \frac{X_f}{R} \quad \text{or} \quad X_f = Q_s R = (10)(10\Omega) = 100\Omega \\
\]

\[ c. \quad L = \frac{X_f}{2\pi f} = \frac{100\Omega}{2\pi(4000 \text{ Hz})} = 3.98 \text{ Hz} \\
X_f = \frac{1}{2\pi f} \quad \text{or} \quad C = \frac{1}{2\pi f} X_f = \frac{1}{2\pi(4000 \text{ Hz})}X_f = 397.89 \text{ nF} \]

**Ex. 20-3**

A series R-L-C circuit has a series resonant frequency of 12 kHz.

a. If \( R = 5\Omega \), and if \( X_f \) at resonance is 300\Omega , find the bandwidth.

c. Find the cutoff frequencies.

\[ a. \quad Q_s = \frac{X_f}{R} = \frac{300\Omega}{5\Omega} = 60 \quad \text{and} \quad BW = \frac{f_2}{Q_s} = \frac{12000 \text{ Hz}}{60} \\
\]

\[ b. \quad Since \ Q_s = 60, \ the \ bandwidth \ is \ bisected \ by \ f_1. \ Therefore, \]

\[ f_2 = f_1 + BW = \frac{12000 \text{ Hz} + 1000 \text{ Hz}}{2} = 12,100 \text{ Hz} \\
and \ f_1 = f_0 - BW = \frac{12,000 \text{ Hz} - 100 \text{ Hz}}{2} = 11,900 \text{ Hz} \]
Ex. 20-4
a. Determine the $Q_s$ and bandwidth for the response curve in Fig. 20.20.
b. For $C = 101.5$ nF, determine $L$ and $R$ for the series resonant circuit.
c. Determine the applied voltage.

Ex. 20-5
A series R-L-C circuit is designed to resonate at $\omega_r = 105$ rad/s, have a bandwidth of 0.15$\omega_r$, and draw 16 W from a 120 V source at resonance.
a. Determine the value of $R$.
b. Find bandwidth in hertz.
c. Find the nameplate values of $L$ and $C$.
d. Determine the $Q_s$ of the circuit.
e. Determine the fractional bandwidth.

HW 20-11 A series resonant circuit is to resonate at $\omega_s = 2\pi \times 10^6$ rad/s and draw 20 W from a 120 V source at resonance. If the fractional bandwidth is 0.16.
a. Determine the resonant frequency in hertz.
b. Calculate the bandwidth in hertz.
c. Determine the values of $R$, $L$, and $C$.
d. Find the resistance of the coil if $Q_s = 80$.

d. $Q_s = \frac{X_L}{R} = 80 \Rightarrow R = \frac{X_L}{80} = 80 \Omega$

e. $Q_s = \frac{X_L}{R} = 80 \Rightarrow R = \frac{X_L}{80} = 80 \Omega$
Parallel Resonance

Electrical and Telecommunications Engineering Technology Department

Professor Jang


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I want to express my gratitude to Prentice Hall giving me the permission to use instructor’s material for developing this module. I would like to thank the Department of Electrical and Telecommunications Engineering Technology of NYCT for giving me support to commence and complete this module. I hope this module is helpful to enhance our students’ academic performance.

Sunghoon Jang

OUTLINES

- Introduction to Parallel Resonance
- Parallel Resonance Circuit
- Unity Power Factor ($f_p$)
- Selectivity Curve
- Effect of $Q_L \geq 10$
- Examples

Key Words: Resonance, Unity Power Factor, Selective Curve, Quality Factor

Parallel Resonance Circuit - Introduction

The basic format of the series resonant circuit is a series R-L-C combination in series with an applied voltage source. The parallel resonant circuit has the basic configuration in Fig. 20.21, a parallel R-L-C combination in parallel with an applied current source.

If the practical equivalent in Fig. 20.22 had the format in Fig. 20.21, the analysis would be as direct and lucid as that experience for series resonance. However, in the practical world, the internal resistance of the coil must be placed in series with the inductor, as shown in Fig. 20.22.
The first effort is to find a parallel network equivalent for the series R-L branch in Fig. 20.22 using the technique in earlier section. That is

\[
Z_{eq} = \frac{1}{\frac{1}{R} + \frac{1}{X_L}} = \frac{1}{\frac{1}{R} + \frac{1}{\omega L}} = \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{2\pi f L}}} = \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{Q}}} = \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{Q}}} = \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{Q}}} = \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{Q}}}
\]

and

\[
Y_{eq} = \frac{1}{\frac{1}{X_L}} = \frac{1}{\frac{1}{\omega L}} = \frac{1}{\frac{1}{\frac{1}{2\pi f L}}} = \frac{1}{\frac{1}{\frac{1}{Q}}} = \frac{1}{\frac{1}{\frac{1}{Q}}} = \frac{1}{\frac{1}{\frac{1}{Q}}} = \frac{1}{\frac{1}{\frac{1}{Q}}}
\]

with

\[
X_L = \frac{1}{\frac{1}{Q}}
\]

as shown in Fig. 20.23.

**Parallel Resonant Circuit - Unity Power Factor, \( f_p \)**

For the network in Fig. 20.25,

\[
\frac{1}{1} \frac{Z_{eq}}{Z_L} + \frac{1}{1} \frac{1}{X_L} + \frac{1}{1} \frac{1}{\frac{1}{2\pi f L}} = \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{Q}}}
\]

and

\[
Y_{eq} = \frac{1}{\frac{1}{X_L}} = \frac{1}{\frac{1}{\frac{1}{2\pi f L}}} = \frac{1}{\frac{1}{\frac{1}{Q}}} = \frac{1}{\frac{1}{\frac{1}{Q}}} = \frac{1}{\frac{1}{\frac{1}{Q}}} = \frac{1}{\frac{1}{\frac{1}{Q}}}
\]

\[= \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{Q}}} \]

Substituting \( R = R_s//R_p \) for the network in Fig. 20.24.

**Parallel Resonant Circuit - Maximum Impedance, \( f_m \)**

At \( f = f_m \), the input impedance of a parallel resonant circuit will be near its maximum value but not quite its maximum value due to the frequency dependence of \( R_s \). The frequency at which impedance occurs is defined by \( f_m \) and is slightly more than \( f_p \) as demonstrated in Fig. 20.26.

The frequency \( f_m \) is determined by differentiating the general equation for \( Z_T \) with respect to frequency and then determining the frequency at which the resulting equation is equal to zero. The resulting equation, however, is the following:

\[ f_m = f \left[ \frac{1 - \frac{1}{4\pi^2 f^2 L^2}}{R + \frac{1}{Q}} \right] \]

Note the similarities with Eq. (20.31). Since square root factor of Eq. (20.32) is always more than the similar factor of Eq. (20.31), \( f_m \) is always closer to \( f \) and more than \( f_p \). In general,

\[ f_m > f > f_p \]

Once \( f_m \) is determined, the network in Fig. 20.25 can be used to determine the magnitude and phase angle of the total impedance at the resonance condition simply by substituting \( f = f_m \) and performing the required calculations. That is

\[ Z_{eq} = R_s//X_L + X_L \]

\[ X_L = \frac{1}{\frac{1}{Q}} \]

**Parallel Resonant Circuit - Selectivity Curve**

Since the current \( I \) of the current source is constant for any value of \( Z_L \) or frequency, the voltage across the parallel circuit will have the same shape as the total impedance \( Z_{eq} \), as shown in Fig. 20.27. For parallel circuit the resonance curve of interest in \( V_c \) derives from electronic considerations that often place the capacitor at the input to another stage of a network.

\[ V_{eff}(f) = I(f) \times Z_{eq}(f) \]

Since the voltage across parallel elements is the same,

\[ V_c = V_p = IZ_c \]

The resonant value of \( V_c \) is therefore determined by the value of \( Z_{eq} \) and magnitude of the current source \( I \). The quality factor of the parallel resonant circuit continues to be determined as following:

\[ Q_p = \frac{R}{X_L} \]

For the ideal current source \( (R_s = \infty) \) or when \( R_s \) is sufficiently large compared to \( R_p \), we can make the following approximation:

\[ Q_p = \frac{X_L}{R_p} \]

In general, the **bandwidth** is still related to the resonant frequency and the quality factor by

\[ BW = f_2 - f_1 = f_p \]

The **cutoff frequencies** \( f_1 \) and \( f_2 \) can be determined using the equivalent network and the quality factor by

\[ f_1 = \frac{1}{4\pi^2} \left( \frac{1}{2\pi f L} + \frac{1}{R} \right) \]

\[ f_2 = \frac{1}{4\pi^2} \left( \frac{1}{2\pi f L} + \frac{1}{R} \right) \]

Where \( f_p \) is the resonant frequency of a parallel resonant circuit (for \( Q_p = 1 \)) and \( f_p \) is the resonant frequency as determined by \( X_L = X_c \) for series resonance. Note that unlike a series resonant circuit, the resonant frequency \( f_p \) is a function of resistance (in this case \( R_s \)).
The effect of $R_L$, $L$, and $C$ on the shape of the parallel resonance curve, as shown in Fig. 20.28 for the input impedance, is quite similar to their effect on the series resonance curve. Whether or not $R_L$ is zero, the parallel resonant circuit frequently appears in a network schematic as shown in Fig. 20.28. At resonance, an increase in $R_L$ or decrease in $\frac{L}{R}$ results in a decrease in the resonant impedance, with a corresponding increase in the current.

### Parallel Resonant Circuit - Effect of $Q_L \geq 10$

The analysis of parallel resonant circuits is significantly more complex than encountered for series circuits. However, this is not the case since, for the majority of parallel resonant circuits, $Q_L$ is sufficiently large to permit a number of approximations that simplify the required analysis.

**Effect of $Q_L \geq 10$ - Inductive Resistance, $X_{lp}$**

$$X_{lp} \approx X_L \quad Q \geq 10 \quad \text{and} \quad X_L \approx X_C \quad Q \geq 10$$

**$Z_T$**

$$Z_T \equiv R_L / R_p = R_L / Q^2 R_L \quad Q \geq 10 \quad \text{and} \quad Q_p \equiv X_L / R_L \quad Q \geq 10, R >> R_L$$

**Q**

$$Q_p = \frac{R}{X_{lp}} \equiv \frac{R}{Q^2 R_L} \quad X_L \quad \text{and} \quad Q \equiv X_L / R_L \quad Q \geq 10, R >> R_L$$

**$BW$**

$$BW = f_2 - f_1 = \frac{f_p}{Q_p} \equiv \frac{1}{2\pi} \frac{R}{L + \frac{1}{RC}} \quad \text{and} \quad BW = f_2 - f_1 \equiv \frac{R}{2\pi} \quad R >> R_L$$

**$I_L$ and $I_C$**

A portion of Fig. 20.30 is reproduced in Fig. 20.31, with $f_p$ defined as shown.

$$I_C \equiv \frac{Q_p}{I_T} \quad Q \geq 10 \quad \text{and} \quad I_L \equiv \frac{Q_L}{I_T} \quad Q \geq 10$$

**Example 20.6**

Given the parallel network in Fig. 20.32 composed of “ideal” elements:

- a. Determine the resonant frequency $f_p$.
- b. Find the total impedance at resonance.
- c. Calculate the quality factor, bandwidth, and cutoff frequencies $f_1$ and $f_2$ of the system.
- d. Find the voltage $V_C$ at resonance.
- e. Determine the currents $I_L$ and $I_C$ at resonance.

- a. The fact that $R_L$ is zero ohms results in a very high $Q_L$ ($= X_L / R_L$), permitting the use of the following equation for $f_p$.

$$f_p = \frac{1}{2\pi \sqrt{LC}} \approx \frac{1}{2\pi} \frac{100}{2 \times 10^6 \pi} \approx 0.16 \text{ kHz}$$

- b. For the parallel reactive elements:

$$Z_{lp} \equiv R_L / R_p = R_L / (2\pi f_0 L) \quad \text{and} \quad BW = 2\pi f_0 L \approx 15.90 \text{ kHz}$$

- c. Establishing the relationship between $I_L$ and $I_C$ at resonance.

$$Z_p = R_p = Q_L R_L$$

- d. The voltage $V_C$ at resonance.

$$V_C = \frac{100}{2 \times 10^6 \pi} \approx 0.0016 \text{ V}$$

- e. Determine the currents $I_L$ and $I_C$ at resonance.

$$I_L = \frac{100}{2 \times 10^6 \pi} \approx 0.0016 \text{ A}$$

$$f_1 = \frac{1}{4\pi} \frac{1}{R} \quad f_2 = \frac{1}{4\pi} \frac{1}{R} \quad = 5.05 \text{ kHz}$$

$$BW = \frac{1}{Q} \approx 5.05 \text{ kHz}$$

**Figure 20.30** Establishing the relationship between $I_L$ and $I_C$ at resonance.

**Figure 20.31** Establishing the relationship between $X_{lp}$ and $X_L$ at resonance.

**Figure 20.32** Example 20.6.
Ex. 20-7 For the parallel resonant circuit in Fig. 20.33 with \( R_s = \infty \Omega \):

a. Determine \( f_s, f_m, \) and \( f_p \), and compare their levels.

b. Calculate the maximum impedance and the magnitude of the voltage \( V_C \) at \( f_m \).

c. Determine the quality factor \( Q_p \).

d. Calculate the bandwidth.

e. Compare the above results with those obtained using the equations associated with \( Q_l \).

Ex. 20-8 For the network in Fig. 20.34 with \( f_p \) provided:

a. Determine \( Q_p \).

b. Determine \( R_p \).

c. Calculate \( Z_m \).

d. Calculate \( Q_p \).

e. Compare the above results with those obtained using the equations associated with \( Q_l \).

Ex. 20-9 Repeat Example 20.9, but ignore the effects of \( R_s \) and compare results.

a. \( f_p \), is the same, 318.31 kHz.

b. For \( R_s = 0 \Omega \),
\[ Q_p = Q_f = 100 \quad (\text{versus} \ 4.76) \]

c. \( BW = f_m - f_s = 3.18 \text{kHz} \) \( (\text{versus} \ 66.87 \text{kHz}) \)

d. \( Z_F = R_p = 1 \Omega \) \( (\text{versus} \ 47.62 \Omega) \)

e. \( V_F = I_F = 2 \text{mA} \times (1 \text{mA}) = 2000 \text{V} \) \( (\text{versus} \ 95.24 \text{V}) \)

Ex. 20-10 Repeat Example 20.9, but ignore the effects of \( R_s \) and compare results.

a. \( f_p \), is the same, 318.31 kHz.

b. For \( R_s = 0 \Omega \),
\[ Q_p = Q_f = 100 \quad (\text{versus} \ 4.76) \]

c. \( BW = f_m - f_s = 3.18 \text{kHz} \) \( (\text{versus} \ 66.87 \text{kHz}) \)

d. \( Z_F = R_p = 1 \Omega \) \( (\text{versus} \ 47.62 \Omega) \)

e. \( V_F = I_F = 2 \text{mA} \times (1 \text{mA}) = 2000 \text{V} \) \( (\text{versus} \ 95.24 \text{V}) \)

HW 20-13 For the “ideal” parallel resonant circuit in Fig. 20.52:

a. Determine the resonant frequency \( f_a \).

b. Find the voltage \( V_F \) at resonance.

c. Determine the currents \( I_L \) and \( I_C \) at resonance.

d. Find \( Q_p \).

Ex. 20-13 Determine \( Q_p \).

\[ Q_p = 2 \omega L \quad \text{versus} \quad Q_p = 2 \omega C \]

Ex. 20-14 Determine \( Q_p \).

\[ Q_p = 2 \omega L \quad \text{versus} \quad Q_p = 2 \omega C \]

Figure 20.33 Example 20.7.

Figure 20.34 Example 20.8.

Figure 20.52 Problem 13.
ETT1222/ET242 Circuit Analysis II

Transformers

Electrical and Telecommunications Engineering Technology Department


Professor Jang

OUTLINES

- Introduction to Transformers
- Mutual Inductance
- The Iron-Core Transformer
- Reflected Impedance and Power

Key Words: Transformer, Mutual Inductance, Coupling Coefficient, Reflected Impedance

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Sunghoon Jang

Transformers - Introduction

Mutual inductance is a phenomenon basic to the operation of the transformer, an electrical device used today in almost every field of electrical engineering. This device plays an integral part in power distribution systems and can be found in many electronic circuits and measuring instruments. In this module, we discuss three of the basic applications of a transformer: to build up or step down the voltage or current, to act as an impedance matching device, and to isolate one portion of a circuit from another.

Transformers - Mutual Inductance

A transformer is constructed of two coils placed so that the changing flux developed by one links the other, as shown in Fig. 22.1. This results in an induced voltage across each coil. To distinguish between the coils, we apply the transformer convention that the coil to which the source is applied is called the primary, and the coil to which the load is applied is called the secondary.
For the primary of the transformer in Fig. 22.1, an application of Faraday’s law result in
\[ e_p = N_p \frac{d\Phi_p}{dt} \text{ (volts, V)} \]
reducing that the voltage induced across the primary is directly related to the number of turns in the primary and the rate of change of magnetic flux linking the primary coil.

\[ e_p = L_p \frac{di_p}{dt} \text{ (volts, V)} \quad (22.2) \]

reducing the induced voltage across the primary is also directly related to the self-inductance of the primary and rate of change of current through the primary winding. The magnitude of \( e_p \), the voltage induced across the secondary, is determined by
\[ e_s = N_p \frac{d\Phi_s}{dt} \text{ (volts, V)} \]

Where \( N_s \) is the number of turns in the secondary winding and \( \Phi_s \) is the portion of primary flux \( \Phi_p \) that links the secondary, then
\[ \Phi_s = \Phi_p \]

and
\[ e_s = M \frac{di_p}{dt} \text{ (volts, V)} \]

The coefficient of coupling (k) between two coil is determined by
\[ k = \frac{e_p}{N_p \frac{d\Phi_p}{dt}} \]

Since the maximum level of \( \Phi_m \) is \( \Phi_p \), the coefficient of coupling between two coils can never be greater than 1. In Fig. 22.2(a), the ferromagnetic steel core ensures that most of the flux linking the primary also links the secondary, establishing a coupling coefficient very close to 1. In Fig. 22.2(b), the fact that both coils are overlapping results in the coil linking the other coil, with the result that the coefficient of coupling is again very close to 1. In Fig. 22.2(c), the absence of a ferromagnetic core results in low levels of flux linkage between the coils. For the secondary, we have

\[ e_s = kN_p \frac{d\Phi_p}{dt} \text{ (volts, V)} \]

The mutual inductance between the two coils in Fig. 22.1 is determined by

\[ M = N_s \frac{d\Phi_s}{dt} \text{ (henries, H)} \]

where

\[ M = N_p \frac{d\Phi_p}{dt} \text{ (henries, H)} \]

Note in the above equations that the symbol for mutual inductance is the symbol for current. The mutual inductance between two coils is proportional to the instantaneous change in flux linking one coil due to an instantaneous change in current through the other coil.

The effective value of \( e_s \) is

\[ E_s = 4.44fN_s \Phi_m \]

which is an equation for the rms value of the voltage across the primary coil in terms of the frequency of the input current or voltage, the number turns of the primary, and the maximum value of the magnetic flux linking the primary.

The flux linking the secondary is

\[ \Phi_s = \frac{E_s}{4.44fN_s} \]

Revealing an important relationship for transformers:

The ratio of the magnitudes of the induced voltages is the same as the ratio of the corresponding turns.

The coefficient of coupling between various coils is indicated in Fig. 22.2. In Fig. 22.2(a), the ferromagnetic steel core ensures that most of the flux linking the primary also links the secondary, establishing a coupling coefficient very close to 1. In Fig. 22.2(b), the fact that both coils are overlapping results in the coil linking the other coil, with the result that the coefficient of coupling is again very close to 1. In Fig. 22.2(c), the absence of a ferromagnetic core results in low levels of flux linkage between the coils. For the secondary, we have

\[ e_s = kN_p \frac{d\Phi_p}{dt} \text{ (volts, V)} \]

The mutual inductance between the two coils in Fig. 22.1 is determined by

\[ M = N_s \frac{d\Phi_s}{dt} \text{ (henries, H)} \]

where

\[ M = N_p \frac{d\Phi_p}{dt} \text{ (henries, H)} \]

In terms of the inductance of each coil and the coefficient of coupling, the mutual inductance is determined by

\[ M = k\sqrt{L_pL_s} \text{ (henries, H)} \]

The greater the coefficient of coupling, or the greater the inductance of either coil, the higher the mutual inductance if we rewrite Eq. (22.3) as

\[ M = N_s \frac{d\Phi_s}{dt} \]

and, since \( M = N_p \frac{d\Phi_p}{dt} \), it can also be written

\[ e_s = M \frac{di_p}{dt} \text{ (volts, V)} \quad \text{and} \quad e_s = M \frac{di_p}{dt} \text{ (volts, V)} \]

Ex. 22-1 For the transformer in Fig. 22.3:

a. Find the mutual inductance \( M \).

b. Find the induced voltage \( e_p \) if the flux \( \Phi_p \) changes at the rate of 450 mWb/s.

c. Find the induced voltage \( e_p \) for the same rate of change indicated in part (b).

d. Find the induced voltages \( e_p \) and \( e_s \) if the current \( i_p \) changes at the rate of 0.2 A/ms.

a. \( M = \sqrt{L_pL_s} = 0.6\sqrt{50\times10^6\times60} = 240 \text{ mH} \)

b. \( e_p = N_p \frac{d\Phi_p}{dt} = (50)(450\times10^6)(22.5) = 48V \)

c. \( e_p = kN_p \frac{d\Phi_p}{dt} = (0.6)(50)(450\times10^6)(22.5) = 27V \)

d. \( e_p = L_p \frac{di_p}{dt} = (200\times10^6)(0.2) = 40V \)

Revealing an important relationship for transformers:

The ratio of the magnitudes of the induced voltages is the same as the ratio of the corresponding turns.

The effective value of \( e_s \) is

\[ E_s = 4.44fN_s \Phi_m \]

which is an equation for the rms value of the voltage across the primary coil in terms of the frequency of the input current or voltage, the number turns of the primary, and the maximum value of the magnetic flux linking the primary.

The flux linking the secondary is

\[ \Phi_s = \frac{E_s}{4.44fN_s} \]

Revealing an important relationship for transformers:

The ratio of the magnitudes of the induced voltages is the same as the ratio of the corresponding turns.

The ratio \( N_s / N_p \), \( a \), is referred to as the transformation ratio: \( a = N_s / N_p \).

If \( a < 1 \), the transformer is called a step-up transformer and if \( a > 1 \), the transformer is called a step-down transformer.
Ex. 22-2 For the iron-core transformer in Fig. 22.5:

(a) Find the maximum flux \( \Phi_m \).

(b) Find the secondary turn \( N_s \).

\[ E_v = 4.44N_f \Phi \]  
Therefore,  
\[ \Phi = \frac{E_v}{4.44N_f} = \frac{200V}{(4.44)(50\Omega)(60Hz)} = 15.02m\Phi \]

(b) \[ E_v = \frac{N_s}{N_p} \]  
Therefore,  
\[ N_s = \frac{N_vE_v}{E_s} = \frac{(50\Omega)(2400V)}{200V} = 600 \text{ turns} \]

The induced voltage across the secondary of the transformer in Fig. 22.4 establish a current \( i_s \) through the load \( Z_s \) and the secondary windings. This current and the turns \( N_s \) develop an mmf \( N_sI_s \) that are not present under no-load conditions since \( i_s = 0 \) and \( N_sI_s = 0 \).

Since the instantaneous values of \( i_p \) and \( i_s \) are related by the turns ratio, the phasor quantities \( I_p \) and \( I_s \) are also related by the same ratio:

\[ N_pI_p = N_sI_s \text{ or } \frac{I_s}{I_p} = \frac{N_p}{N_s} \]

The primary and secondary currents of a transformer are therefore related by the inverse ratios of the turns.

Ex. 22-3 For the iron-core transformer in Fig. 22.6:

(a) Find the magnitude of the current in the primary and the impressed voltage across the primary.

(b) Find the input resistance of the transformer.

\[ I_p = \frac{N_s}{N_p} \]
\[ V_p = I_pZ_p = \frac{N_s}{N_p} \frac{V_s}{N_s} = \frac{V_s}{N_p} \]
\[ \frac{V_p}{V_s} = \frac{N_s}{N_p} \]

\[ V_p = \frac{N_s}{N_p} \cdot 200V = 1600V \]

\[ Z_p = a^2Z_s \]
\[ a = \frac{N_s}{N_p} \]
\[ Z_p = (8)^2(2\Omega) = 128\Omega \]

Transformers - Reflected Impedance and Power

In previous section we found that  
\[ \frac{V_s}{V_p} = \frac{N_s}{N_p} = a \text{ and } \frac{I_p}{I_s} = \frac{N_s}{N_p} = \frac{1}{a} \]

Dividing the first by the second, we have  
\[ \frac{V_p}{I_p} = \frac{I_s}{V_s} \text{ or } \frac{V_p}{I_p} = a^2 \text{ and } \frac{I_p}{I_s} = \frac{V_s}{V_p} = a^2 \]

However, since  
\[ Z_p = \frac{V_p}{I_p} \text{ and } Z_s = \frac{V_s}{I_s} \]

then  
\[ Z_p = a^2Z_s \]

That is, the impedance of the primary circuit of an ideal transformer is the transformation ratio squared times the impedance of the load. Note that if the load is capacitive or inductive, the reflected impedance is also capacitive or inductive. For the ideal iron-core transformer,  
\[ \frac{E_p}{E_s} = \frac{I_p}{I_s} \text{ or } \frac{E_p}{I_p} = \frac{E_s}{I_s} \]

and  
\[ P_e = P_i \text{ (ideal condition) } \]

HW 12-13

(a) If \( N_p = 400 V, V_s = 1200 \), and \( V_p = 100 V \), find the magnitude of \( I_p \) for the iron-core transformer in Fig. 22.5 if \( Z_s = 9\Omega + j12\Omega \).

(b) Find the magnitude of the voltage \( V_L \) and the current \( I_L \) for the conditions of part (a).

\[ a = \frac{N_p}{N_s} = \frac{400\Omega}{1200\Omega} = \frac{1}{3} \]
\[ Z_s = a^2Z_p = \left( \frac{1}{3} \right)^2 \left( 9\Omega + j12\Omega \right) = \frac{1}{3} \Omega + \frac{j1}{3} \Omega = 1.667\Omega \angle 53.13^\circ \]
\[ I_p = \frac{V_p}{Z_s} = \frac{100V}{1.667\Omega} = 60A \]
\[ V_L = I_LZ_s = \left( \frac{1}{3} \right)(20A)(15\Omega) = 300V \]

Homework 22: 1-3,4,8,12